STRICT COMPARISON OF POSITIVE ELEMENTS IN MULTIPLIER ALGEBRAS

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Abstract. Main result: If a C*-algebra $\mathcal{A}$ is simple, σ-unital, has finitely many extremal traces, and has strict comparison of positive elements by traces, then its multiplier $\mathcal{M}(\mathcal{A})$ also has strict comparison of positive elements by traces. The same results holds if "finitely many extremal traces" is replaced by "quasicontinuous scale".

A key ingredient in the proof is that every positive element in the multiplier algebra of an arbitrary σ-unital C*-algebra can be approximated by a bi-diagonal series. An application of strict comparison: If $\mathcal{A}$ is a simple separable stable σ-unital C*-algebra with real rank zero, stable rank one, strict comparison of positive elements by traces, then whether a positive element is a linear combination of projections depends on the trace values of its range projection.

1. Introduction

The notion of strict comparison of positive elements in a C*-algebra plays an important role in Cuntz semigroups and has attracted an increasing interest in recent years. For instance in [29, Corollary 4.6] Rordam has proven that if $\mathcal{A}$ is unital simple exact finite and $\mathcal{Z}$-stable, then $\mathcal{A}$ has strict comparison of positive elements by traces. It was conjectured in 2008 by Toms and Winter that for separable simple nuclear C*-algebras, $\mathcal{Z}$-stability is equivalent to strict comparison of positive elements by traces (see also [22] [31]).

All unital, simple, exact, finite , and $\mathcal{Z}$-stable C*-algebras have strict comparison of positive elements by traces. This large class of C*-algebras includes irrational rotation algebra, higher dimensional simple noncommutative tori, crossed products of minimal homeomorphisms on compact metric spaces with finite covering dimension, simple unital AH-algebras with bounded dimension growth, the Jiang-Su algebra and many others.

In a previous paper we have proven that if $\mathcal{A}$ is a unital separable simple nonelementary C*-algebra with real rank zero, and has finitely many extremal traces, and strict comparison of projections by traces, then $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has strict comparison of projections by traces provided that the definition is appropriately adapted to the presence of ideals in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ ([17, Theorem 3.2]).

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The main goal this paper is to extend this result [17, Theorem 3.2] to a larger class of algebras and to strict comparison of positive elements. We will prove in Theorem 5.4 that under the assumption that a simple σ-unital C*-algebra $\mathcal{A}$ has only finitely many extremal traces and has strict comparison of positive elements by traces, then the property of strict comparison of positive elements by traces holds also in the multiplier algebra $\mathcal{M}(\mathcal{A})$.

The condition that the extremal boundary is finite can be replaced by the weaker condition that the algebra has quasicontinuous scale (Theorem 6.6), but cannot be removed completely. Indeed in a subsequent paper ([18]) we prove that strict comparison for the multiplier algebra can fail when the extremal boundary is infinite.

Our original motivation for obtaining strict comparison of positive elements by traces in the multiplier algebra was to apply them to the problem of decomposing positive elements into *positive combination of projections* (PCP for short), namely into sums $\sum_1^n \lambda_j p_j$ where p_j are projections in $\mathcal{A}$, λ_j are positive scalars, and n is a finite integer.

In [11] and [15] we investigated the notion of PCP in the setting of purely infinite C*-algebras and W*-algebras respectively (see also [12], [13], [14]). Focusing then on finite algebras, we proved in [16, Theorem 6.1] that if $\mathcal{A}$ is a simple separable stable σ-unital C*-algebra with real rank zero, stable rank one, strict comparison of projections by traces and has finitely many extremal traces, then $a \in \mathcal{A}_+$ is a PCP if and only if $\tau(R_a) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$, where R_a denotes the range projection. A key ingredient in the proof was Brown's interpolation theorem [2].

When the multiplier algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ of an algebra $\mathcal{A}$ as above has real rank zero (and thus Brown's interpolation theorem is again available) a similar result holds for $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$: a necessary and sufficient condition for $A \in (\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))_+$ to be a PCP is that either $\tau(R_A) < \infty$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which A belongs to the trace ideal I_τ or A is a full element. ([17, Theorem 6.4]).

Strict comparison of positive elements permits us to obtain in Theorem 7.9 precisely the same result but dropping the hypothesis that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is of real rank zero.

Our paper is organized as follows.

In the preliminary section §2 we review the notions of tracial simplex for non-unital C*-algebras, strict comparison of of positive elements by traces, and prove some lemmas on Cuntz subequivalence which we need for this paper. In section §3 we obtain some technical results on convergence of dimension functions of cut-offs of strictly converging monotone sequences (Lemma 3.2).

The main tool in the proof of our main result, but possibly also of independent interest, is Theorem 4.2 where we prove that positive elements in the multiplier algebra of an arbitrary σ-unital C*-algebra can be approximated by a bi-diagonal series (see Definition 4.1). This is an extension and improvement to arbitrary σ-unital C*-algebras of the tri-diagonal form obtained previously by Elliott [6] for AF algebras and by Zhang [32, Theorem 2.2] for real rank zero C*-algebras.

Based on the above results, in section §5 we present the proof of strict comparison in the multiplier algebra (Theorem 5.4) broken into a couple of technical lemmas.

In section §6 we then extend this result to the case where $\mathcal{A}$ has a quasicontinuous scale (see Definition 6.1 and Theorem 6.6).

In section §7 we apply strict comparison of positive elements in the multiplier algebra to the problem of decomposing positive elements into positive combination

of projections. The proof of Theorem 7.9 is based on several steps, some of which may have independent interest.

In Proposition 7.7 we use strict comparison of positive elements to show that every positive element $A \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})_+$ majorizes a scalar multiple of a projection P that generates the same ideal as A.

A second step is the extension and reformulation of the "2×2" Lemma 7.2 which played a key role in obtaining PCP decompositions in purely C*-algebras and in W*-algebras (see [11], [15].) This lemma also provides bounds on the number of projections needed for a PCP decomposition.

The third essential tool is given by Lemma 7.3, where we show, roughly speaking, that every σ-unital hereditary sub-algebras of $\mathcal{M}(\mathcal{A})$ is *-isomorphic to a hereditary subalgebra of a unital corner of the multiplier algebra.

2. Preliminaries

2.1. **Pedersen ideal and approximate identities.** For a simple C*-algebra $\mathcal{A}$ the Pedersen ideal $\mathrm{Ped}(\mathcal{A})$ is the minimal dense ideal of $\mathcal{A}$ ([24], [21]). It contains all the positive elements with a local unit, that is the elements $a \in \mathcal{A}_+$ for which there exists $b \in \mathcal{A}_+$ such that $ba = a$. In fact

$$(\mathrm{Ped}(\mathcal{A}))_+ = \{x \in \mathcal{A}_+ \mid x \le \sum_1^n y_j \quad \text{for some } n \in \mathbb{N},\ y_j \in \mathcal{A}_+ \text{ with local unit.}\}$$

Let $\mathcal{B}$ be a σ-unital hereditary sub-algebra of $\mathcal{A}$ let h be a strictly positive element of $\mathcal{B}$ with $\|h\| = 1$, and let $e_n := f_{\frac{1}{n}}(h)$ where

$$f_\epsilon(t) = \begin{cases} 0 & \text{for } t \in [0, \frac{\epsilon}{1+\epsilon}] \\ \frac{1+\epsilon}{\epsilon^2}t - \frac{1}{\epsilon} & \text{for } t \in (\frac{\epsilon}{1+\epsilon}, \epsilon) \\ 1 & \text{for } t \in [\epsilon, 1]. \end{cases} \tag{2.1}$$

It is well known and routine to verify that $\{e_n\}_1^\infty$ is an approximate identity of $\mathcal{B}$ satisfying (2.2)

$$e_{n+1}e_n = e_n \quad \forall\, n \tag{2.2}$$

and $e_n \in \mathrm{Ped}(\mathcal{A})$ for all n.

2.2. **Traces and dimension functions.** For a simple C*-algebra we denote by $\mathcal{T}(\mathcal{A})$ the collection of the (norm) lower semicontinuous densely defined tracial weights on $\mathcal{A}_+$, henceforth, traces for short. Explicitly, a trace τ

- is an additive and homogeneous map from $\mathcal{A}_+$ into $[0, \infty]$ (a weight);
- satisfies the trace condition $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{A}$;
- the cone $\{x \in \mathcal{A}_+ \mid \tau(x) < \infty\}$ is norm dense in $\mathcal{A}_+$ (thus τ is also called densely finite, or semifinite);
- satisfies the condition $\tau(x) \le \underline{\lim}\, \tau(x_n)$ for $x, x_n \in \mathcal{A}_+$ and $\|x_n - x\| \to 0$, or equivalently, $\tau(x) = \lim \tau(x_n)$ for $0 \le x_n \uparrow x$ in norm.

Recall that every trace is finite on $\mathrm{Ped}(\mathcal{A})$, and hence $\tau(e_n) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$.

Using the notations in [30], for every $0 \ne e \in \mathrm{Ped}(\mathcal{A})_+$) set

$$\mathcal{T}(\mathcal{A})_{e\mapsto 1} := \{\tau \in \mathcal{T}(\mathcal{A}) \mid \tau(e) = 1\}. \tag{2.3}$$

Then $\mathcal{T}(\mathcal{A})_{e\mapsto 1}$ is a cone base for $\mathcal{T}(\mathcal{A})$ and can be viewed as a normalization (or scale) of $\mathcal{T}(\mathcal{A})$. When equipped with the topology of pointwise convergence on $\mathrm{Ped}(\mathcal{A})$, $\mathcal{T}(\mathcal{A})_{e\mapsto 1}$ is a Choquet simplex ([30, Proposition 3.4]). Set

$$\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A})) \text{ to be the collection of the extreme points of } \mathcal{T}(\mathcal{A})\,.$$

We call $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ the *extremal boundary* of $\mathcal{T}(\mathcal{A})$ and its elements *extreme traces.*

Given two nonzero elements $e, f \in \mathrm{Ped}(\mathcal{A})_+$, the natural one-to-one map

$$\mathcal{T}(\mathcal{A})_{e\mapsto 1} \ni \tau \mapsto \frac{1}{\tau(f)}\tau \in \mathcal{T}(\mathcal{A})_{f\mapsto 1}$$

is a homeomorphism which maps faces onto faces and in particular, extreme points onto extreme points. Thus the cardinality of $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ does not depend on the normalization chosen.

To simplify notations, we will henceforth identify $\mathcal{T}(\mathcal{A})$ with $\mathcal{T}(\mathcal{A})_{e\mapsto 1}$. (For more details, see [30], [7], and also [16]).

If $\mathcal{A}$ is unital, then $\mathrm{Ped}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{T}(\mathcal{A})_{1\mapsto 1}$ coincides with the usual tracial state simplex. Thus the definition of $\mathcal{T}(\mathcal{A})$ that we use coincides with the standard one when $\mathcal{A}$ is unital, and hence, by Brown's Stabilization theorem [1] also when $\mathcal{A}$ is stable and has a nonzero projection p.

Furthermore, as remarked in [16, 5.3], by the work of F. Combes [4, Proposition 4.1 and Proposition 4.4] and Ortega, Rordam, and Thiel [23, Proposition 5.2] every $\tau \in \mathcal{T}(\mathcal{A})$ has a unique extension, still denoted by τ, to a lower semicontinuous (i.e., normal) tracial weight (trace for short) on the enveloping von Neumann algebra $\mathcal{A}^{**}$.

As usual, the dimension function $d_\tau(\cdot)$ is defined on $\mathcal{M}(\mathcal{A})_+$ as

$$d_\tau(A) =: \lim_n \tau(A^{1/n}) \quad \forall\; A \in \mathcal{M}(\mathcal{A})_+,\; \tau \in \mathcal{T}(\mathcal{A})\,.$$

As shown in [23, Remark 5.3], $d_\tau(A) = \tau(R_A)$ where R_A is the range projection of A. In particular

$$d_\tau\Big((A-\delta)_+\Big) = \tau(R_{(A-\delta)_+}) = \tau(\chi_{(\delta,\|A\|]}(A)) \quad \forall\; \delta \geq 0. \tag{2.4}$$

We will also recall that for all $0 \neq A \in \mathcal{M}(\mathcal{A})_+$ both the maps

$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto d_\tau(A) \in [0,\infty] \tag{2.5}$$

and

$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto \tau(A) = \hat{A}(\tau) \in [0,\infty] \tag{2.6}$$

are affine, lower semicontinuous, and strictly positive.

2.3. **Cuntz subequivalence.** Let $\mathcal{A}$ be a C*-algebra. If p, q are projections in $\mathcal{A}$, $p \sim q$ (resp, $p \preceq q$) denotes the Murray - von Neumann equivalence, (resp., subequivalence) that is $p = vv^*,\ q = v^*v$ for some $v \in \mathcal{A}$ (resp. $p \sim p' \leq q$ for some projection $p' \in \mathcal{A}$).

If $a, b \in \mathcal{A}_+$, $a \preceq b$ denotes the Cuntz sub-equivalence of positive elements, that $\|a - x_n b x_n^*\| \to 0$ for some sequence $x_n \in \mathcal{A}$. For ease of reference we list here the following known facts (e.g., see [28]).

Lemma 2.1. *Let $\mathcal{A}$ be a C*-algebra, $a, b \in \mathcal{A}_+$, $\delta > 0$. Then*
(i) *If $a \leq b$ then $a \preceq b$.*
(ii) *If $\|a - b\| < \delta$ then $(a-\delta)_+ \preceq b$.*

(iii) *If* $a \preceq b$*, then there is* $r \in \mathcal{A}$ *such that* $(a-\delta)_+ = rbr^*$.
(iv) *If* $a \preceq b$*, then there is* $\delta' > 0$ *and* $r \in \mathcal{A}$ *such that* $(a-\delta)_+ = r(b-\delta')_+ r^*$.
(v) $a + b \preceq a \oplus b$.
(vi) *If* $a \preceq b$ *then* $d_\tau(a) \leq d_\tau(b)$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$.
(vii) $\tau(b) \leq \|b\| d_\tau(b)$ *and* $d_\tau((b-\delta)_+) < \frac{1}{\delta}\tau(b)$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$.

We will need an adaptation of [20, Lemma 1.1].

Lemma 2.2. *Let* $\mathcal{A}$ *be a* C^**-algebra,* $a, b \in \mathcal{A}_+$*, and* $\delta > 0$*. If* $a \preceq (b-\delta)_+$*, then for every* $\epsilon > 0$*,* $(a-\epsilon)_+ = xbx^*$ *for some* $x \in \mathcal{A}$ *with* $\|x\|^2 \leq \frac{\|a\|}{\delta}$*. Furthermore,* x *can be chosen so that* $xx^* \leq c_1(a-\epsilon)_+$ *and* $x^*x \leq c_2(b-\delta)_+$ *for some constants* c_1 *and* c_2*.*

Proof. As a consequence of [28, Proposition 2.4], there is an $s \in \mathcal{A}$ for which

$$(a-\epsilon)_+ = s(b-\delta)_+ s^*.$$

Then

$$\|s(b-\delta)_+^{1/2}\| = \|(a-\epsilon)_+\|^{1/2} \leq \|a\|^{1/2}.$$

Let

$$h_\epsilon(t) = \begin{cases} \frac{t}{\epsilon} & t \in [0,\epsilon] \\ 1 & t \in [\epsilon, \|a\|] \end{cases} \quad \text{and} \quad g_\delta(t) = \begin{cases} \frac{1}{\delta} & t \in [0,\delta] \\ \frac{1}{t} & t \in [\delta, \|b\|] \end{cases}.$$

Then both functions are continuous and

$$\|h_\epsilon(a)\| = 1, \quad (a-\epsilon)_+ = h_\epsilon(a)(a-\epsilon)_+,$$
$$\|g_\delta(b)\| = \frac{1}{\delta}, \quad (b-\delta)_+ = g_\delta(b) b (b-\delta)_+.$$

Set

$$x = h_\epsilon(a) s (b-\delta)_+^{1/2} g_\delta^{1/2}(b).$$

Then

$$\begin{aligned} xbx^* &= h_\epsilon(a) s (b-\delta)_+^{1/2} g_\delta^{1/2}(b) b g_\delta^{1/2}(b)(b-\delta)_+^{1/2} s^* h_\epsilon(a) \\ &= h_\epsilon(a) s (b-\delta)_+ s^* h_\epsilon(a) \\ &= h_\epsilon(a)(a-\epsilon)_+ h_\epsilon(a) \\ &= (a-\epsilon)_+. \end{aligned}$$

Moreover,

$$\|x\| \leq \|h_\epsilon(a)\| \|s(b-\delta)_+^{1/2}\| \, \|g_\delta(b)^{1/2}\| \leq \|a\|^{1/2} \frac{1}{\delta^{1/2}},$$

$$\begin{aligned} xx^* &= h_\epsilon(a) s (b-\delta)_+^{1/2} g_\delta(b)(b-\delta)_+^{1/2} s^* h_\epsilon(a) \\ &\leq \frac{1}{\delta} h_\epsilon(a) s (b-\delta)_+ s^* h_\epsilon(a) = \frac{1}{\delta} h_\epsilon(a)(a-\epsilon)_+ h_\epsilon(a) \\ &= \frac{1}{\delta}(a-\epsilon)_+, \end{aligned}$$

and

$$\begin{aligned} x^*x &= g_\delta(b)^{1/2}(b-\delta)_+^{1/2} s^* h_\epsilon(a)^2 s (b-\delta)_+^{1/2} g_\delta^{1/2}(b) \\ &\leq \|s\|^2 g_\delta(b)^{1/2}(b-\delta)_+ g_\delta^{1/2}(b) \leq \frac{\|s\|^2}{\delta}(b-\delta)_+. \end{aligned}$$

□

Notice that if $a, b \in \mathcal{A}$ are selfadjoint and $a \leq b$, in general it does not follow that $a_+ \leq b_+$. However, we often need less.

Lemma 2.3. *Let $\mathcal{A}$ be a C^*-algebra and $a, b \in \mathcal{A}$ be selfadjoint. If $a \leq b$ then $a_+ \preceq b_+$.*

Proof. Since $a \leq b \leq b_+$ and since $\delta(t-\delta)_+ \leq t(t-\delta)_+$ for all t and $\delta > 0$, then

$$(a-\delta)_+ \leq \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} a \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} \leq \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} b_+ \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} \preceq b_+.$$

As a consequence, $(a-\delta)_+ \preceq b_+$ for all δ and hence $a_+ \preceq b_+$. □

Lemma 2.4. *Let $\mathcal{A}$ be a C^*-algebra, $a, b \in A_+$, $\delta_i \geq 0$ with $\delta_1 \geq \delta_2 + \delta_3$, then*
(i) $(a+b-\delta_1)_+ \preceq (a-\delta_2)_+ + (b-\delta_3)_+$
(ii) $d_\tau(a+b) \leq d_\tau(a) + d_\tau(b)$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$
(iii) $d_\tau\big((a+b-\delta_1)_+\big) \leq d_\tau\big((a-\delta_2)_+\big) + d_\tau\big((b-\delta_3)_+\big)$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$.

Proof.
(i) Without loss of generality, $\delta_1 = \delta_2 + \delta_3$ Then

$$a + b - \delta_1 = (a-\delta_2) + (b-\delta_3) \leq (a-\delta_2)_+ + (b-\delta_3)_+$$

hence the conclusion follows from Lemma 2.3.
(ii) Well known
(iii) This follows from (i), the monotonicity of d_τ with respect to $\preceq$, and (ii). □

The following simple fact will be used in section §7.

Lemma 2.5. *Let $\mathcal{A}$ be a C^*-algebra, $a \in \mathcal{A}_+$, $q \in \mathcal{A}$ be a projection, and $\delta > 0$ a real number. If $q \preceq (a-\delta)_+$, then there is a projection $p \sim q$ such that $a \geq \delta p$. If $a \geq \delta p$ for some projection p then $p \preceq (a-\delta')_+$ for all $0 \leq \delta' < \delta$.*

Proof. Assume that $q \preceq (a-\delta)_+$. Since by Lemma 2.1 (v),

$$\frac{1}{2} q = (q - \frac{1}{2})_+ = x(a-\delta)_+ x^* \quad \text{for some} \quad x \in \mathcal{A},$$

it follows that $q = (\sqrt{2}x)(a-\delta)_+(\sqrt{2}x)^*$. Thus

$$q \sim p := (a-\delta)_+^{\frac{1}{2}} 2x^*x (a-\delta)_+^{\frac{1}{2}} \leq 2\|x\|^2 (a-\delta)_+$$

Then $p \leq R_{(a-\delta)_+} = \chi_{(\delta, \|a\|]}(a) \leq \frac{1}{\delta} a$. Assume now that $a \geq \delta p$ and $0 \leq \delta' < \delta$. then $a - \delta' \geq (\delta - \delta')p - \delta' p^\perp$ hence by Lemma 2.3

$$(\delta - \delta')p = \big((\delta-\delta')p - \delta' p^\perp\big)_+ \preceq (a-\delta')_+$$

and hence $p \preceq (a-\delta')_+$. □

2.4. **Strict comparison.** When $\mathcal{A}$ is a simple stably finite C*-algebra, we say that $\mathcal{A}$ has strict comparison of positive elements by traces if $\mathcal{T}(\mathcal{A}) \neq \emptyset$ and if $a, b \in \mathcal{A}_+$ and $d_\tau(a) < d_\tau(b)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$, then $a \preceq b$. Notice that this definition is weaker than the standard definition that requires the same property for all elements in $\mathcal{A} \otimes \mathcal{K}_+$ (or sometimes for all elements of $M_n(\mathcal{A})_+$ and all $n \in \mathbb{N}$), but this weaker property is all we need for Theorem 5.4.

Clearly, this definition would be vacuous for $\mathcal{M}(\mathcal{A})$ (when $\mathcal{A}$ is not unital) if there is any element $b \in \mathcal{A}_+$ such that $d_\tau(b) = \infty$ for all τ (which is always the case when $\mathcal{A}$ is stable). Indeed every element $A \in \mathcal{M}(\mathcal{A})$, including full elements,

would then satisfy the condition that $d_\tau(A) < d_\tau(b)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$. On the other hand, the condition $A \preceq b$ implies that A belongs to the ideal generated by b, which is $\mathcal{A}$ by the assumption that $\mathcal{A}$ is simple. To avoid this obvious obstruction we use the following definition where $I(B)$ denotes the ideal of $\mathcal{M}(\mathcal{A})$ generated by B.

Definition 2.6. *Let $\mathcal{A}$ be a simple C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. We say that $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces if $A \preceq B$ for $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$.*

2.5. **Ideals in $\mathcal{M}(\mathcal{A})$.** For every $\tau \in \mathcal{T}(\mathcal{A})$

$$K_\tau := \{A \in \mathcal{M}(\mathcal{A})_+ \mid \tau(A) < \infty\}$$

is a hereditary cone of $\mathcal{M}(\mathcal{A})_+$ which by the trace property satisfies the condition that if $X^*X \in K_\tau$, then $XX^* \in K_\tau$. Let

$$L(K_\tau) := \{X \in \mathcal{M}(\mathcal{A}) \mid X^*X \in K_\tau\}$$

be the associated two-sided ideal of $\mathcal{M}(\mathcal{A})$ and let

$$I_\tau := \overline{L(K_\tau)}.$$

Then it is immediate to see from [25, Theorem 1.5.2] that

$$I_\tau := \overline{\{X \in \mathcal{M}(\mathcal{A}) \mid \tau(X^*X) < \infty\}} = \overline{\operatorname{span}\{K_\tau\}}$$

where the closures are in norm. The following is also well known (for a proof see for instance [17, Lemma 2.6])

$$A \in (I_\tau)_+ \text{ if and only } d_\tau\big((A-\delta)_+\big) < \infty \text{ for every } \delta > 0. \tag{2.7}$$

3. Dimension functions of cut-offs of monotone sequences

Lemma 3.1. *Let $\mathcal{A}$ be a C^*-algebra, T_n, T be normal elements of $M(\mathcal{A})$, and $K \subseteq \mathbb{C}$ be a compact set for which the spectrum $\sigma(T_n)$ is contained in K for all n and $\sigma(T) \subseteq K$, and assume that $T_n \to T$ strictly. Then*

$$f(T_n) \to f(T) \quad \textit{strictly}$$

for every continuous function $f : K \to \mathbb{C}$.

Proof. This is immediate when f is a polynomial in one complex variable. Then apply Stone–Weierstrass Theorem. □

Lemma 3.2. *Let $\mathcal{A}$ be σ-unital C^*-algebra, $\tau \in \mathcal{T}(\mathcal{A})$, $T_n, T \in \mathcal{M}(\mathcal{A})_+$ and assume that $T_n \to T$ in the strict topology. Then*

(i) *If $T_n \le T_{n+1}$ for all n, then $d_\tau\big((T_n-\delta)_+\big) \uparrow d_\tau\big((T-\delta)_+\big)$ for all $\delta \ge 0$.*
(ii) *If $T = 0$, $T_n \ge T_{n+1}$ for all n, and $T_1 \in I_\tau$, then $d_\tau\big((T_n-\delta)_+\big) \downarrow 0$ for all $\delta > 0$.*
(iii) *If $T_n \ge T_{n+1}$ for all n and $T_1 \in I_\tau$, then for all $0 < \epsilon < \delta$*

$$d_\tau\big((T-\delta)_+\big) \le \lim_n d_\tau\big((T_n-\delta)_+\big) \le d_\tau\big((T-\delta+\epsilon)_+\big).$$

Proof. Assume without loss of generality that $\|T\| \le 1$. Since strict convergence implies strong convergence in the enveloping W*-algebra, it is easy to verify that in case (i) $T_n \le T$ and in case (ii) $T_n \ge T$.

(i) Since $T_n - \delta \le T_{n+1} - \delta \le T - \delta$ for every n and hence, by Lemma 2.3,

$$(T_n - \delta)_+ \preceq (T_{n+1} - \delta)_+ \preceq (T - \delta)_+$$

it follows that

$$d_\tau((T_n - \delta)_+) \le d_\tau((T_{n+1} - \delta)_+) \le d_\tau((T - \delta)_+)$$

and hence

$$\lim_n d_\tau((T_n - \delta)_+) \le d_\tau((T - \delta)_+). \tag{3.1}$$

Now we prove the opposite inequality.

Since $\mathcal{A}$ is σ-unital, there is an approximate identity of $\mathcal{A}$ consisting of an increasing sequence e_n such that $e_{n+1}e_n = e_n$ for all n. As $T_n \to T$ strictly and since $\sigma(T), \sigma(T_n) \subset [0,1]$ for all n, by Lemma 3.1, it follows that for every $N \in \mathbb{N}$,

$$\begin{aligned} &(T_n - \delta)_+^{1/N} \to (T - \delta)_+^{1/N} && \text{strictly} \\ &\lim_n e_k^{1/2}(T_n - \delta)_+^{1/N} e_k^{1/2} = e_k^{1/2}(T - \delta)_+^{1/N} e_k^{1/2} && \text{in norm.} \end{aligned}$$

Now τ is norm continuous on $e_k^{1/2}\mathcal{M}(\mathcal{A})e_k^{1/2} = e_k^{1/2}\mathcal{A}e_k^{1/2}$ because $e_k \in \mathrm{Ped}(\mathcal{A})$ and hence $\tau(e_k) < \infty$. As a consequence,

$$\lim_n \tau\big(e_k^{1/2}(T_n - \delta)_+^{1/N} e_k^{1/2}\big) = \tau\big(e_k^{1/2}(T - \delta)_+^{1/N} e_k^{1/2}\big) \tag{3.2}$$

and thus

$$\begin{aligned} \tau\Big(\big((T-\delta)_+\big)^{1/N}\Big) &= \lim_k \tau\Big(\big((T - \delta)_+\big)^{1/2N} e_k \big((T - \delta)_+\big)^{1/2N}\Big) && \text{(normality of } \tau) \\ &= \lim_k \tau\Big(e_k^{1/2}\big((T - \delta)_+\big)^{1/N} e_k^{1/2}\Big) && \text{(trace property)} \\ &= \lim_k \lim_n \tau\Big(e_k^{1/2}\big((T_n - \delta)_+\big)^{1/N} e_k^{1/2}\Big) && \text{(by (3.2))} \\ &= \lim_k \lim_n \tau\Big(\big((T_n - \delta)_+\big)^{1/2N} e_k (T_n - \delta)_+\big)^{1/2N}\Big) && \text{(trace property)} \\ &\le \underline{\lim}_n \tau\Big(\big((T_n - \delta)_+\big)^{1/N}\Big) && \text{(monotonicity of } \tau) \\ &\le \underline{\lim}_n d_\tau\big((T_n - \delta)_+\big) && (\text{as } \|(T_n - \delta)_+\| \le 1) \\ &= \lim_n d_\tau\big((T_n - \delta)_+\big) && (\text{as } d_\tau\big((T_n - \delta)_+\big) \uparrow). \end{aligned}$$

It follows that

$$\lim_n d_\tau\big((T_n - \delta)_+\big) \ge \lim_N \tau\Big(\big((T - \delta)_+\big)^{1/N}\Big) = d_\tau\big((T - \delta)_+\big)$$

and equality follows from (3.1).

(ii) Let $\epsilon > 0$ and let $Q_n := \chi_{(\delta,\infty)}(T_n)$, $P_\epsilon := \chi_{(\epsilon,\infty)}(T_1^{1/2})$. These spectral projections belong to the von Neumann algebra $\mathcal{A}^{**}$ and commute with T_n and T_1 respectively. Recall that we identify every $\tau \in \mathcal{T}(\mathcal{A})$ with its extension to $\mathcal{A}^{**}$ ([23, Proposition 5.2], see also §2.2) and that the trace of the range projection of a positive operator is just the dimension function of that operator. In particular,

$$\tau(Q_n) = d_\tau((T_n - \delta)_+) \le d_\tau((T_1 - \delta)_+) = \tau(Q_1) < \infty \tag{3.3}$$

and

$$d_\tau((T_n - \delta)_+) \le \frac{1}{\delta}\tau(T_nQ_n) = \frac{1}{\delta}\Big(\tau\big(P_\epsilon(T_nQ_n)P_\epsilon\big) + \tau\big(P_\epsilon^\perp(T_nQ_n)P_\epsilon^\perp\big)\Big). \tag{3.4}$$

Since $T_1 \in I_\tau$, it follows that $\tau(P_\epsilon) = d_\tau((T_1 - \epsilon)_+) < \infty$ and hence τ is σ-weak continuous on $P_\epsilon \mathcal{A}^{**} P_\epsilon$. Therefore

$$\tau\big(P_\epsilon(T_n Q_n)P_\epsilon\big) \leq \tau\big(P_\epsilon T_n P_\epsilon\big) \to 0. \tag{3.5}$$

Since $T_n \leq T_1$, there are $G_n \in \mathcal{A}^{**}$ such that $T_n^{1/2} = G_n T_1^{1/2} = T_1^{1/2} G_n^*$ and $\|G_n\| \leq 1$. Then

$$\|P_\epsilon^\perp T_n^{1/2}\| = \|\chi_{[0,\epsilon)}(T_1^{1/2})T_1^{1/2}G_n^*\| \leq \epsilon.$$

From here and (3.3) we have

$$\tau\big(P_\epsilon^\perp T_n Q_n P_\epsilon^\perp)\big) = \tau\big(Q_n T_n^{1/2} P_\epsilon^\perp T_n^{1/2} Q_n\big) \leq \epsilon^2 \tau(Q_n) \leq \epsilon^2 \tau(Q_1).$$

Thus by (3.5) and (3.4), it follows that $d_\tau((T_n - \delta)_+) \to 0$.

(iii) By the same argument as in part (i), $d_\tau\big((T_n - \delta)_+\big) \downarrow$ and hence

$$\lim_n d_\tau\big((T_n - \delta)_+\big) \geq d_\tau\big((T - \delta)_+\big).$$

By Lemma 2.4 (iii), for every $0 < \epsilon < \delta$

$$d_\tau\big((T_n - \delta)_+\big) \leq d_\tau\big((T - \delta + \epsilon)_+\big) + d_\tau\big((T_n - T - \epsilon)_+\big)$$

By part (ii), $\lim_n d_\tau\big((T_n - T - \epsilon)_+\big) = 0$, which concludes the proof. □

Remark 3.3. *Unlike in (i), for part (ii) we need to assume that $\delta > 0$. Indeed, let P be a projections with $0 < \tau(P) < \infty$. Then $T_n := \frac{1}{n}P \downarrow 0$ in norm, yet $d_\tau(T_n) \equiv \tau(P) \not\to 0$.*
Similarly, in (iii) we need to assume that $\epsilon > 0$. Indeed let as above P be a projections with $0 < \tau(P) < \infty$. Then $T_n := (\delta + \frac{1}{n})P \downarrow \delta P = T$ in norm, yet $d_\tau\big((T_n - \delta)_+\big) \equiv \tau(P)$ while $d_\tau\big((T - \delta)_+\big) = 0$.

For ease of use in the following section, let us single out the following special case.

Corollary 3.4. *Let $\mathcal{A}$ be σ-unital C^*-algebra, $\tau \in \mathcal{T}(\mathcal{A})$, and assume that $D := \sum_1^\infty d_k \in \mathcal{M}(\mathcal{A})$ is the sum of a series of elements $d_k \in \mathcal{A}_+$ converging in the strict topology. Then*

(i) $\lim_n d_\tau\Big(\big(\sum_{i=m}^n d_i - \delta\big)_+\Big) = d_\tau\Big(\big(\sum_{i=m}^\infty a_i - \delta\big)_+\Big)$ *for every $\delta \geq 0$ and $m \in \mathbb{N}$.*

(ii) *If $D \in I_\tau$ then $\lim_n d_\tau\Big(\big(\sum_{i=n}^\infty d_i - \delta\big)_+\Big) = 0$ for every $\delta > 0$.*

4. Bi-diagonal decomposition

The following theorem uses a modification of the proof for Theorem 2.2 of [32] to decompose any positive element in a general σ-unital C*-algebras into the sum of a "bi-diagonal" series and "small" remainder in $\mathcal{A}$. Notice that there is no need to assume the existence of an approximate identity of projections. By bi-diagonal we mean the following:

Definition 4.1. *Let $\mathcal{A}$ be a σ-unital C^*-algebra and let $d_k \in \mathcal{A}_+$. We say that the series $D := \sum_1^\infty d_k$ is bi-diagonal if $\sum_1^\infty d_k$ converges in the strict topology and $d_n d_m = 0$ for $|n - m| \geq 2$.*

Every bi-diagonal series $\sum_1^\infty d_k$ can thus written as a sum of two diagonal series, namely $\sum_1^\infty d_{2k}$, and $\sum_1^\infty d_{2k+1}$, but the sum of two diagonal series is not necessarily bi-diagonal.

Theorem 4.2. *Let $\mathcal{A}$ be a σ-unital C^*-algebra and let $T \in \mathcal{M}(\mathcal{A})_+$. Then for every $\epsilon > 0$ there exist a bi-diagonal series $\sum_1^\infty d_k$ and a self-adjoint element $a_\epsilon \in \mathcal{A}$ with $\|a_\epsilon\| < \epsilon$ such that $T = \sum_1^\infty d_k + a_\epsilon$. Furthermore, the elements $d_k \in \mathrm{Ped}(\mathcal{A})$ and for a fixed approximate identity $\{e_n\}$ of $\mathcal{A}$ with $e_{n+1}e_n = e_n$, for every $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ for which $e_n \sum_N^\infty d_k = 0$.*

Proof. Let $\{e_n\}$ be an increasing approximate identity of $\mathcal{A}$ and as usual we assume that $e_{n+1}e_n = e_n$ and set $e_0 = 0$ (see (2.2)). As a consequence

$$(e_n - e_{n-1})(e_m - e_{m-1}) = 0 \quad \forall\, |n - m| \geq 2.$$

Assume without loss of generality that $\|T\| = 1$ and let $a_k := T^{1/2}(e_k - e_{k-1})T^{1/2}$. Then $a_k \in \mathcal{A}_+$ for all k and $T = \sum_1^\infty a_k$ where the series converges strictly. We will construct inductively two strictly increasing sequences of positive integers $\{m_k\}_0^\infty$ and $\{n_k\}_1^\infty$ as follows. Start by setting $m_0 = n_0 = 0$ and $m_1 = 1$ and choosing $n_1 \geq 1$ such that

$$\|a_1 - e_{n_1} a_1 e_{n_1}\| < \frac{\epsilon}{2} \qquad \text{since } e_n \to 1 \text{ strictly and } a_1 \in \mathcal{A}.$$

Now choose $m_2 > m_1$ and $n_2 > n_1$ such that

$$\|e_{n_1} \sum_{j=m_2+1}^\infty a_j\| < \left(\frac{\epsilon}{2^5}\right)^2 \qquad (\text{since } \sum_{j=m}^\infty a_j \to 0 \text{ strictly and } e_{n_1} \in \mathcal{A})$$

$$\|(1 - e_{n_2}) \sum_{j=m_1+1}^{m_2} a_j\| < \frac{\epsilon}{2^4} \qquad (\text{since } e_n \to 1 \text{ strictly and } \sum_{j=m_1+1}^{m_2} a_j \in \mathcal{A}).$$

Set $b_1 := a_1$ and iterate the construction:

$$\text{choose } m_k \text{ so that} \quad \|e_{n_{k-1}} \sum_{j=m_k+1}^\infty a_j\| < \left(\frac{\epsilon}{2^{k+3}}\right)^2 \tag{4.1}$$

$$\text{set} \quad b_k := \sum_{j=m_{k-1}+1}^{m_k} a_j \tag{4.2}$$

$$\text{choose } n_k \text{ so that} \quad \|(1 - e_{n_k})b_k\| < \frac{\epsilon}{2^{k+2}}. \tag{4.3}$$

Since e_{n_k} is also an approximate unit, to simplify notations assume henceforth that $n_k = k$. Set for all $k \geq 1$

$$c_1 := e_1 b_1 e_1$$
$$c_k := (e_k - e_{k-2}) b_k (e_k - e_{k-2}) \qquad \forall\, k \geq 2.$$

From (4.1) (applied to $k - 1$) we see that

$$\begin{aligned} \|e_{k-2} b_k\| &\leq \|e_{k-2} b_k^{1/2}\| = \|e_{k-2} b_k e_{k-2}\|^{1/2} \\ &\leq \|e_{k-2} \sum_{j=m_{k-1}+1}^\infty a_j e_{k-2}\|^{1/2} \\ &\leq \|e_{k-2} \sum_{j=m_{k-1}+1}^\infty a_j\|^{1/2} < \frac{\epsilon}{2^{k+2}}. \end{aligned}$$

From the decomposition

$$b_k - c_k = (1 - e_k)b_k + e_k b_k (1 - e_k) + e_k b_k e_{k-2} + e_{k-2} b_k (e_k - e_{k-2})$$

and from the above inequality and (4.3) we thus obtain that

$$\|b_k - c_k\| < \frac{\epsilon}{2^k} \qquad \forall\, k.$$

As a consequence the series $a_\epsilon := \sum_{k=1}^\infty (b_k - c_k)$ converges in norm and hence $a_\epsilon = a_\epsilon^* \in \mathcal{A}$. Since $T = \sum_{k=1}^\infty a_k = \sum_{k=1}^\infty b_k$, the series $\sum_{k=1}^\infty b_k$ converges strictly and hence so does the series $D := \sum_{k=1}^\infty c_k = T - a_\epsilon$. Now set

$$d_k := c_{2k-1} + c_{2k} \quad \forall\, k \geq 1.$$

Then $D = \sum_{k=1}^\infty d_k$ and

$$\begin{aligned} d_1 &= c_1 + c_2 = e_1 b_1 e_1 + e_2 b_2 e_2 \in e_2 \mathcal{A} e_2 \\ d_k &= (e_{2k-1} - e_{2k-3}) b_{2k-1} (e_{2k-1} - e_{2k-3}) + (e_{2k} - e_{2k-2}) b_{2k} (e_{2k} - e_{2k-2}). \end{aligned}$$

As a consequence, $d_n d_m = 0$ for all $|n - m| \geq 2$.

By construction, all the elements d_k have a local unit and it is immediate to verify that $e_n \sum_N^\infty d_k = 0$ for every $N \geq \frac{n+4}{2}$. □

The method of the proof of Theorem 4.2 can be applied to give a joint bi-diagonal form to multiple elements in $\mathcal{M}(\mathcal{A})$ in the following sense:

Remark 4.3. *Let $\mathcal{A}$ be a σ-unital C^*-algebra and let $T_1, T_2, \cdots T_N \in \mathcal{M}(\mathcal{A})_+$. Then for every $\epsilon > 0$ there exist N bi-diagonal series $\sum_{k=1}^\infty d_{k,j}$ and self-adjoint elements $a_{\epsilon,j} \in \mathcal{A}$, $1 \leq j \leq N$ with $\|a_{\epsilon,j}\| < \epsilon$ such that $T_j = \sum_1^\infty d_{k,j} + a_{\epsilon,j}$ and $d_{n,i} d_{m,j} = 0$ for $|n - m| \geq 2$ and all $1 \leq i, j \leq N$.*

Thus, up to a small remainder, every element in $\mathcal{M}(\mathcal{A})_+$ is bi-diagonal and hence the sum of two diagonal series. Diagonal series are used extensively in multiplier algebras. We will need the following result relating Cuntz subequivalence of (cut-offs of) summands in two diagonal series to Cuntz subequivalence of (cut-offs of) their sums. Notice that we do not need to require that the summands belong to $\mathcal{A}$.

Proposition 4.4. *Let $\mathcal{A}$ be a C^*-algebra, $A = \sum_1^\infty A_n$, $B = \sum_1^\infty B_n$ where $A_n, B_n \in \mathcal{M}(\mathcal{A})_+$, $A_n A_m = 0$, $B_n B_m = 0$ for $n \neq m$ and the two series converge in the strict topology. If $A_n \preceq (B_n - \delta)_+$ for some $\delta > 0$ and for all n, then $A \preceq (B - \delta')_+$ for all $0 < \delta' < \delta$.*

Proof. By Lemma 2.2 applied to $A_n \preceq \Big((B_n - \delta') - (\delta - \delta')\Big)_+$, for every n there is an $X_n \in \mathcal{A}$ such that

$$\begin{aligned} (A_n - \epsilon)_+ &= X_n (B_n - \delta') X_n^*, \\ \|X_n\|^2 &\leq \frac{\|A_n\|}{\delta - \delta'} \leq \frac{\sup_n \|A_n\|}{\delta - \delta'}, \\ X_n X_n^* &\leq c_{1,n} (A_n - \epsilon)_+, \\ X_n^* X_n &\leq c_{2,n} (B_n - \delta)_+. \end{aligned}$$

for some constants $c_{1,n}$ and $c_{2,n}$. Therefore

$$\begin{aligned} X_n X_n^* &\leq \|X_n\|^2 R_{(A_n - \epsilon)_+} \leq \frac{\sup_n \|X_n\|^2}{\epsilon} A_n \\ X_n^* X_n &\leq \|X_n\|^2 R_{(B_n - \delta)_+} \leq \frac{\sup_n \|X_n\|^2}{\delta} B_n. \end{aligned}$$

As a consequence, $R_{X_n} \leq R_{A_n}$ and $R_{X_n^*} \leq R_{B_n}$. But then,

$$R_{X_n} R_{X_m} = R_{X_n^*} R_{X_m^*} = 0 \ \forall n \neq m \quad \text{and hence} \quad X_n X_m^* = X_n^* X_m = 0 \ \forall n \neq m.$$

Thus for every $m < n \in \mathbb{N}$ and $a \in \mathcal{A}$

$$\begin{aligned} \|a \sum_{k=m}^{n} X_k\|^2 &= \|a \sum_{k,k'=m}^{n} X_k X_{k'}^* a^*\| = \|a \sum_{k=m}^{n} X_k X_k^* a^*\| \\ &\leq \frac{\sup_n \|X_n\|^2}{\epsilon} \|a \sum_{k=m}^{n} A_k a^*\| \leq \|a\| \frac{\sup_n \|X_n\|^2}{\epsilon} \|a \sum_{k=m}^{n} A_k\|. \end{aligned}$$

Similarly

$$\| \sum_{k=m}^{n} X_k a\|^2 \leq \|a\| \frac{\sup_n \|X_n\|^2}{\delta} \| \sum_{k=m}^{n} B_k a\|.$$

Since the series $\sum_1^\infty A_n$ and $\sum_1^\infty B_n$ converge strictly, it follows that $\sum_1^\infty X_n$ converges strictly. Let $X := \sum_1^\infty X_n$. Then $X \in \mathcal{M}(\mathcal{A})$ and since $X_n = X_n R_{B_n}$ and $R_{B_n}(B_n - \delta')_+ = (B_n - \delta')_+$ for every n,

$$\begin{aligned} (A - \epsilon)_+ &= \sum_1^\infty (A_n - \epsilon)_+ \\ &= \sum_1^\infty X_n (B_n - \delta')_+ X_n^* \\ &= \Big(\sum_1^\infty X_n \Big) \sum_1^\infty (B_n - \delta')_+ \Big(\sum_1^\infty X_n^* \Big) \\ &= X (B - \delta')_+ X^*. \end{aligned}$$

Since ϵ is arbitrary, it follows that $A \preceq (B - \delta')_+$. □

Remark 4.5. *From the above proof we see that if the series $\sum_1^\infty A_n$ converges in norm, then the series $\sum_1^\infty X_n$ also converges in norm.*

5. Strict comparison of positive elements in $\mathcal{M}(\mathcal{A})$.

For which simple C*-algebras $\mathcal{A}$ does strict comparison of positive elements by traces hold for $\mathcal{M}(\mathcal{A})$ when it holds for $\mathcal{A}$? In this section we prove that a sufficient condition is that $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite. In a subsequent paper we will prove that when $\mathcal{A}$ is stable and contains a nonzero projection, finiteness of $\partial_e(\mathcal{T}(\mathcal{A}))$ is indeed necessary. In fact when $\partial_e(\mathcal{T}(\mathcal{A}))$ is infinite even strict comparison of projections by traces fails ([18, Proposition 4.5]).

We will need the following notation: for every $B \in \mathcal{M}(\mathcal{A})_+$, let

$$F(B) = \mathrm{co}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\} \tag{5.1}$$

denote the convex combination of the extremal traces for which $B \notin I_\tau$ and let $F(B)'$ be its complementary face (the union of the faces disjoint from $F(B)$, so the largest face disjoint from $F(B)$). Either $F(B)$ or $F(B)'$ can be empty. For this and for other basic results on convexity theory and Choquet simplexes we refer the reader to [9]. If $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite, then both $F(B)$ and $F(B)'$ are closed and by [9, Theorem 11.28],

$$\mathcal{T}(\mathcal{A}) = F(B) \dotplus F(B)',$$

is the direct convex sum of $F(B)$ and $F(B)'$, that is, $F(B)\cap F(B)'=\emptyset$ and every $\tau\in\mathcal{T}(\mathcal{A})\setminus(F(B)\cup F(B)'$ has a unique decomposition $\tau=t\mu+(1-t)\mu'$ for some $0<t<1$, $\mu\in F(B)$, and $\mu'\in F(B)'$. Thus

$$F(B)'=\operatorname{co}\{\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\mid B\in I_\tau\}=\{\tau\in\mathcal{T}(\mathcal{A})\mid B\in I_\tau\}. \tag{5.2}$$

Lemma 5.1. *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra with finite extremal tracial boundary $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$. Let $A,B\in\mathcal{M}(\mathcal{A})_+$ such that $A\in I(B)$ and $d_\tau(A)<d_\tau(B)$ for all those $\tau\in\mathcal{T}(\mathcal{A})$ for which $d_\tau(B)<\infty$. Then for every $\epsilon>0$ there are $\delta>0$ and $\alpha>0$ such that*
(i) $d_\tau((A-\epsilon)_+)+\alpha\le d_\tau((B-\delta)_+)<\infty$ *if* $\tau\in F(B)'$
(ii) $d_\tau((B-\delta)_+)=\infty$ *if* $\tau\in F(B)$.

Proof. Notice first that if $d_\tau(B)<\infty$, then $B\in I_\tau$, i.e., $\tau\in F(B)'$.
If $d_\tau(B)=\infty$ for all $\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$, set $\alpha:=1$, otherwise, set

$$\alpha:=\frac{1}{2}\min\big\{d_\tau(B)-d_\tau\Big(A\Big)\mid\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A})),\ d_\tau(B)<\infty\big\}.$$

Then it is easy to see that then

$$d_\tau(A)+2\alpha\le d_\tau(B)\quad\forall\,\tau\in\mathcal{T}(\mathcal{A})\,. \tag{5.3}$$

By (2.7) we have

$$d_\tau\big((B-\delta)_+\big)\begin{cases}<\infty & \forall\tau\in F(B)',\forall\delta>0\\ =\infty & \forall\tau\in F(B),\exists\delta>0.\end{cases} \tag{5.4}$$

Since $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ is finite, we can choose $\delta_o>0$ such that $d_\tau\big((B-\delta_0)_+\big)=\infty$ for all $\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\cap F(B)$ and hence for all $\tau\in F(B)$.
Since $(B-\delta)_+\uparrow B$ for $\delta\downarrow 0$ (convergence in norm), it follows that

$$d_\tau\big((B-\delta)_+\big)\uparrow d_\tau(B)\quad\forall\,\tau\in\mathcal{T}(\mathcal{A})\,.$$

By hypothesis and (5.2), $A\in I_\tau$ for all $\tau\in F(B)'$, hence by (2.7), $d_\tau\big((A-\epsilon)_+\big)<\infty$. Then for every $\tau\in F(B)'\cap\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ we can choose $0<\delta_\tau\le\delta_o$ such that

$$d_\tau\big((B-\delta_\tau)_+\big)\ge\begin{cases}d_\tau(B)-\alpha & \text{if } d_\tau(B)<\infty\\ d_\tau\big((A-\epsilon)_+\big)+\alpha & \text{if } d_\tau(B)=\infty.\end{cases}$$

Since by (5.3)

$$d_\tau(B)-\alpha\ge d_\tau(A)+\alpha\ge d_\tau\big((A-\epsilon)_+\big)+\alpha\quad\forall\,\tau\in\mathcal{T}(\mathcal{A}),$$

and since $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ is finite, by choosing $\delta=\min\{\delta_\tau\mid\tau\in F(B)'\cap\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\}$, we have

$$d_\tau\big((B-\delta)_+\big)\ge d_\tau\big((A-\epsilon)_+\big)+\alpha\quad\forall\,\tau\in F(B)'\cap\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\,.$$

It is then immediate to see that the same inequality holds for all $\tau\in F(B)'$. Moreover, $d_\tau\big((B-\delta)_+\big)<\infty$ by (5.4) which proves (i). Finally, (ii) follows also from (5.4) since $\delta\le\delta_o$. ☐

Next is our key lemma which deals with the special case of cut-downs of bi-diagonal series.

Lemma 5.2. *Let $\mathcal{A}$ be a σ-unital nonunital simple C*-algebra with strict comparison of positive elements by traces and with finite extremal tracial boundary $\partial_e(\mathcal{T}(\mathcal{A}))$, $\sum_{i=1}^\infty a_i$, $\sum_{i=1}^\infty b_i$ be two bi-diagonal series in $\mathcal{M}(\mathcal{A})_+$, and let F be a closed face of $\mathcal{T}(\mathcal{A})$ and F' its complementary face (either F or F' can be empty). Assume that for some $\epsilon, \delta, \alpha > 0$ we have*

$$\begin{aligned}
&\big(\textstyle\sum_{i=1}^\infty b_i - \delta\big)_+ \notin \mathcal{A} && \\
&\textstyle\sum_1^\infty a_i \in I_\tau && \text{if } \tau \in F' \\
&d_\tau\big((\textstyle\sum_{i=1}^\infty a_i - \epsilon)_+\big) + \alpha \le d_\tau\big((\textstyle\sum_{i=1}^\infty b_i - \delta)_+\big) < \infty && \text{if } \tau \in F' \\
&d_\tau\big((\textstyle\sum_{i=1}^\infty b_i - \delta)_+\big) = \infty && \text{if } \tau \in F.
\end{aligned}$$

Then for any $0 < \delta' < \delta$, $\epsilon' > \epsilon$

$$\Big(\sum_{i=1}^\infty a_i - 2\epsilon'\Big)_+ \preceq \Big(\sum_{i=1}^\infty b_i - \delta'\Big)_+.$$

Proof. The case when one of the faces F' or F is empty is simpler and is left to the reader, so we assume that both are non-empty.

We construct iteratively a strictly increasing sequence of positive integers m_k and two interlaced sequences of positive integers n_k, n'_k with

$$n_k + 2 \le n'_k \le n_{k+1} - 2 \tag{5.5}$$

First we notice that by Corollary 3.4 (i)

$$d_\tau\Big(\big(\sum_{i=1}^n b_i - \delta\big)_+\Big) \uparrow d_\tau\Big(\big(\sum_{i=1}^\infty b_i - \delta\big)_+\Big)$$

for all $\tau \in \mathcal{T}(\mathcal{A})$. Since $|\,\partial_e(\mathcal{T}(\mathcal{A}))\,| < \infty$, the convergence is uniform on $\mathcal{T}(\mathcal{A})$ and hence on F'. Thus we can choose n_1 so that for all $\tau \in F'$

$$d_\tau\Big(\big(\sum_{i=1}^\infty a_i - \epsilon\big)_+\Big) < d_\tau\Big(\big(\sum_{i=1}^{n_1} b_i - \delta\big)_+\Big). \tag{5.6}$$

Since $\Big(\sum_{i=1}^\infty b_i - \delta\Big)_+ \notin \mathcal{A}$, by Lemma 2.4 (i) it follows that for every $n \in \mathbb{N}$

$$\Big(\sum_{i=1}^\infty b_i - \delta\Big)_+ \preceq \sum_{i=1}^{n-1} b_i + \Big(\sum_{i=n}^\infty b_i - \delta\Big)_+.$$

Since $\sum_{i=1}^{n-1} b_i \in \mathcal{A}$, it follows that $\Big(\sum_{i=n}^\infty b_i - \delta\Big)_+ \neq 0$ and hence by Lemma 3.1

$$\Big(\sum_{i=n}^m b_i - \delta\Big)_+ \to \Big(\sum_{i=n}^\infty b_i - \delta\Big)_+$$

in the strict topology. Thus we can find some $n'_1 \ge n_1 + 2$ so that

$$\Big(\sum_{i=n_1+2}^{n'_1} b_i - \delta\Big)_+ \neq 0. \tag{5.7}$$

Now $d_\tau\Big(\big(\sum_{i=m}^\infty a_i - \epsilon\big)_+\Big) \downarrow 0$ for all $\tau \in F'$ by Corollary 3.4 (ii) and again the convergence is uniform on F'. Thus we can choose m_1 so that for all $\tau \in F'$

$$d_\tau\Big(\big(\sum_{i=m_1+1}^{\infty} a_i - \epsilon\big)_+\Big) < d_\tau\Big(\big(\sum_{i=n_1+2}^{n_1'} b_i - \delta\big)_+\Big). \tag{5.8}$$

Finally, $d_\tau\Big(\big(\sum_{i=n_1'+2}^{n} b_i - \delta\big)_+\Big) \uparrow \infty$ for all $\tau \in F$ by Corollary 3.4 (i). Furthermore, because $|F \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))| < \infty$, we can choose $n_2 \geq n_1' + 2$ so that for all $\tau \in F$

$$d_\tau\Big(\big(\sum_{i=1}^{m_1} a_i - \epsilon\big)_+\Big) < d_\tau\Big(\big(\sum_{i=n_1'+2}^{n_2} b_i - \delta\big)_+\Big). \tag{5.9}$$

Set

$$c_1 := \sum_{i=1}^{m_1} a_i \qquad \text{and} \qquad d_1 := \sum_{i=1}^{n_1} b_i + \sum_{i=n_1'+2}^{n_2} b_i.$$

Notice that

$$\sum_{i=1}^{n_1} b_i \perp \sum_{i=n_1'+2}^{n_2} b_i \tag{5.10}$$

by the condition that $b_i b_j = 0$ for $|i-j| \geq 2$. But then for all $\tau \in F \cup F'$ we have

$$\begin{aligned}
d_\tau\big((c_1-\epsilon)_+\big) &= d_\tau\Big(\big(\sum_{i=1}^{m_1} a_i - \epsilon\big)_+\Big) \\
&< d_\tau\Big(\big(\sum_{i=1}^{n_1} b_i - \delta\big)_+\Big) + d_\tau\Big(\big(\sum_{i=n_1'+2}^{n_2} b_i - \delta\big)_+\Big) \quad \text{(by (5.6), (5.9))} \\
&= d_\tau\Big(\big(\sum_{i=1}^{n_1} b_i + \sum_{i=n_1'+2}^{n_2} b_i - \delta\big)_+\Big) \quad \text{(by (5.10))} \\
&= d_\tau\big((d_1-\delta)_+\big).
\end{aligned}$$

Since every $\tau \in \mathcal{T}(\mathcal{A})$ has a decomposition as a convex combination of elements in F and F' (unique if $\tau \notin F \cup F'$) it follows that

$$d_\tau\big((c_1-\epsilon)_+\big) < d_\tau\big((d_1-\delta)_+\big) \qquad \forall\, \tau \in \mathcal{T}(\mathcal{A}). \tag{5.11}$$

Next choose integers n_2', n_3 and m_2 such that $n_2+2 \leq n_2' \leq n_3 - 2$ and $m_2 > m_1$ and

$$\big(\sum_{i=n_2+2}^{n_2'} b_i - \delta\big)_+ \neq 0 \qquad \text{(as in (5.7))}$$

$$d_\tau\Big(\big(\sum_{i=m_2+1}^{\infty} a_i - \epsilon\big) +\Big) < d_\tau\Big(\big(\sum_{i=n_2+2}^{n_2'} b_i - \delta\big)_+\Big) \quad \tau \in F' \qquad \text{(as in (5.8))}$$

$$d_\tau\Big(\big(\sum_{i=m_1+1}^{m_2} a_i - \epsilon\big)_+\Big) < d_\tau\Big(\big(\sum_{i=n_2'+2}^{n_3} b_i - \delta\big)_+\Big) \quad \tau \in F \qquad \text{(as in (5.9))}.$$

Set

$$c_2 := \sum_{i=m_1+1}^{m_2} a_i \qquad \text{and} \qquad d_2 := \sum_{i=n_1+2}^{n_1'} b_i + \sum_{i=n_2'+2}^{n_3} b_i.$$

Because the summands come from blocks removed by more than one index we have

$$\sum_{i=n_1+2}^{n_1'} b_i \perp \sum_{i=n_2'+2}^{n_3} b_i \quad \text{and} \quad d_1 \perp d_2.$$

Hence as for the first step of the proof we have for all $\tau \in \mathcal{T}(\mathcal{A})$

$$d_\tau\big((c_2-\epsilon)_+\big) < d_\tau\Big(\big(\sum_{i=n_1+2}^{n_1'} b_i - \delta\big)_+\Big) + d_\tau\Big(\big(\sum_{i=n_2'+2}^{n_3} b_i - \delta\big)_+\Big) = d_\tau\big((d_2-\delta)_+\big).$$

Notice that d_2 has a different form than the beginning term d_1, but from $k=3$ on we can iterate the construction keeping the same form as d_2. Thus we choose a strictly increasing sequence of integers m_k and two interlaced sequences n_k, n_k' with

$$n_k + 2 \leq n_k' \leq n_{k+1} - 2$$

so that setting

$$c_k := \sum_{i=m_{k-1}+1}^{m_k} a_i \quad \text{and} \quad d_k := \sum_{i=n_{k-1}+2}^{n_{k-1}'} b_i + \sum_{i=n_k'+2}^{n_{k+1}} b_i.$$

we have $\sum_{i=n_{k-1}+2}^{n_{k-1}'} b_i \perp \sum_{i=n_k'+2}^{n_{k+1}} b_i$ and

$$d_\tau\big((c_k-\epsilon)_+\big) < d_\tau\big((d_k-\delta)_+\big) \qquad \forall\, k \in \mathbb{N},\, \tau \in \mathcal{T}(\mathcal{A})\,. \tag{5.12}$$

By the strict comparison of positive elements of $\mathcal{A}$ by traces, it follows that

$$(c_k-\epsilon)_+ \preceq (d_k-\delta)_+ \quad \forall\, k. \tag{5.13}$$

Moreover, from this construction we see that

$$c_ic_j = 0 \quad |i-j| \geq 2 \quad \text{and} \quad d_id_j = 0 \quad i \neq j. \tag{5.14}$$

Thus by construction,

$$\sum_1^\infty c_{2k} + \sum_1^\infty c_{2k+1} = \sum_1^\infty a_i$$
$$\sum_1^\infty d_k \leq \sum_1^\infty b_i$$

where three series $\sum_1^\infty c_{2k}$, $\sum_1^\infty c_{2k+1}$, and $\sum_1^\infty d_k$ are diagonal (sums of mutually orthogonal terms that converge in the strict topology). Then by (5.13) and Proposition 4.4 we obtain

$$\big(\sum_k c_{2k} - \epsilon'\big)_+ \preceq \Big(\sum_k d_{2k} - \delta'\Big)_+$$
$$\big(\sum_k c_{2k+1} - \epsilon'\big)_+ \preceq \Big(\sum_k d_{2k+1} - \delta'\Big)_+.$$

Finally,

$$\begin{aligned}
(\sum_1^\infty a_i - 2\epsilon')_+ &\preceq \big(\sum_k c_{2k} - \epsilon'\big)_+ + \big(\sum_k c_{2k+1} - \epsilon'\big)_+ && \text{(by Lemma 2.4)}\\
&\preceq \big(\sum_k c_{2k} - \epsilon'\big)_+ \oplus \big(\sum_k c_{2k+1} - \epsilon'\big)_+ && \text{(by Lemma 2.1)}\\
&\preceq \big(\sum_k d_{2k} - \delta'\big)_+ + \big(\sum_k d_{2k+1} - \delta'\big)_+ && \text{(since } \sum_k d_{2k} \perp \sum_k d_{2k+1})\\
&= \big(\sum_k d_k - \delta'\big)_+ && \text{(since } \sum_k d_{2k} \perp \sum_k d_{2k+1})\\
&\preceq \big(\sum_i b_i - \delta'\big)_+ && \text{(since } \sum_k d_k \leq \sum_k b_k).
\end{aligned}$$

□

Notice that in this proof we used the finiteness of $|\,\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\,|$ in two different ways. We used it directly to guarantee the finiteness of $F \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ which was essential in some steps in the proof. However the fact that $F' \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ is finite was used only to guarantee that the pointwise convergence on F' provided by Corollary 3.4 was uniform. Thus the conclusions of this lemma will hold also without the condition that $F' \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ is finite provided that the convergence provided by Corollary 3.4 is uniform on F'. This observation will be used in the next section dealing with C*-algebras with quasi-continuous scale and so we record it as follows:

Lemma 5.3. *Let $\mathcal{A}$ be a σ-unital nonunital simple C*-algebra with strict comparison of positive elements by traces. Let $\sum_{i=1}^\infty a_i$ and $\sum_{i=1}^\infty b_i$ be two bi-diagonal series in $\mathcal{M}(\mathcal{A})_+$. Let F be a face F' be its complementary face (F or F' can be empty). Assume that for some $\epsilon, \delta, \alpha > 0$*

(i) $\big(\sum_{i=1}^\infty b_i - \delta\big)_+ \notin \mathcal{A}$
(ii) $\sum_1^\infty a_i \in I_\tau$ *if* $\tau \in F'$
(iii) $d_\tau\big((\sum_{i=1}^\infty a_i - \epsilon)_+\big) + \alpha \leq d_\tau\big((\sum_{i=1}^\infty b_i - \delta)_+\big) < \infty$ *if* $\tau \in F'$
(iv) $d_\tau\big((\sum_{i=1}^\infty b_i - \delta)_+\big) = \infty$ *if* $\tau \in F$
(v) $F \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ *is finite*
(vi) *for every* $m \in \mathbb{N}$, $d_\tau\Big(\big(\sum_m^n b_j - \delta\big)_+\Big) \uparrow d_\tau\Big(\big(\sum_m^\infty b_j - \delta\big)_+\Big)$ *uniformly on* F'
(vii) $d_\tau\Big(\big(\sum_n^\infty a_j - \epsilon\big)_+\Big) \downarrow 0$ *uniformly on* F'.

Then for any $0 < \delta' < \delta$, $\epsilon' > \epsilon$

$$\Big(\sum_{i=1}^\infty a_i - 2\epsilon'\Big)_+ \preceq \Big(\sum_{i=1}^\infty b_i - \delta'\Big)_+.$$

We are now in position to state and prove our main theorem.

Theorem 5.4. *Let $\mathcal{A}$ be a σ-unital simple C*-algebra with strict comparison of positive elements by traces and with $|\,\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\,| < \infty$. Then strict comparison of positive element by traces holds in $\mathcal{M}(\mathcal{A})$.*

Proof. Let $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Since strict comparison holds on $\mathcal{A}$, we can assume without loss of generality that $B \notin \mathcal{A}$. Let $\epsilon > 0$. By Lemma 5.1 we can choose

$\delta > 0$ and $\alpha > 0$ such that

$$\begin{cases} d_\tau((A-\epsilon)_+) + \alpha \le d_\tau((B-\delta)_+) < \infty & \tau \in F(B)' \\ d_\tau((B-\delta)_+) = \infty & \tau \in F(B). \end{cases} \tag{5.15}$$

By Theorem 4.2 we can find bi-diagonal decompositions

$$A = \sum_{i=1}^{\infty} a_i + a_o \quad \text{and} \quad B = \sum_{i=1}^{\infty} b_i + b_o$$

where the series converge strictly and $a_i, b_i \in \mathcal{A}_+$, (in fact they are in $\mathrm{Ped}(\mathcal{A})$,) $a_i a_j = b_i b_j = 0$ for $|i-j| \ge 2$, $a_o, b_o \in \mathcal{A}_{sa}$, $\|a_o\| < \epsilon$, and $\|b_o\| < \frac{\delta}{3}$.

Since $B - b_o = \sum_{i=1}^{\infty} b_i \notin \mathcal{A}$, and since

$$\Big(\sum_{i=1}^{\infty} b_i - \delta'\Big)_+ \uparrow \sum_{i=1}^{\infty} b_i \quad \text{for} \ \ \delta' \downarrow 0,$$

there is a $\delta_o > 0$ such that $\Big(\sum_{i=1}^{\infty} b_i - \delta'\Big)_+ \notin \mathcal{A}$ for all $\delta' \le \delta_o$. Replacing δ with $\min(\delta_o, \delta)$ does not decrease $d_\tau\big((B-\delta)_+\big)$ for any $\tau \in \mathcal{T}(\mathcal{A})$ and assures that $(\sum_{i=1}^{\infty} b_i - \delta)_+ \notin \mathcal{A}$.

For all $\tau \in F(B)'$ we have $B \in I_\tau$, hence $A \in I_\tau$. Since $a_o \in \mathcal{A} \subset I_\tau$, it follows that $\sum_1^{\infty} a_i = A - a_o \in I_\tau$.

By Lemma 2.1 (ii)

$$\big(\sum_{i=1}^{\infty} a_i - 2\epsilon\big)_+ \preceq (A-\epsilon)_+ \quad \text{and} \quad (B-\delta)_+ \preceq \big(\sum_{i=1}^{\infty} b_i - \frac{2\delta}{3}\big)_+.$$

Hence for all $\tau \in \mathcal{T}(\mathcal{A})$

$$d_\tau\Big(\big(\sum_{i=1}^{\infty} a_i - 2\epsilon\big)_+\Big) \le d_\tau\Big((A-\epsilon)_+\Big) \quad \text{and} \quad d_\tau\Big((B-\delta)_+\Big) \le \Big(\big(\sum_{i=1}^{\infty} b_i - \frac{2\delta}{3}\big)_+\Big).$$

Thus

$$\begin{array}{ll} \big(\sum_{i=1}^{\infty} b_i - \delta\big)_+ \notin \mathcal{A} & \\ \sum_1^{\infty} a_i \in I_\tau & \text{if } \tau \in F(B)' \\ d_\tau\big((\sum_{i=1}^{\infty} a_i - 2\epsilon)_+\big) + \alpha \le d_\tau\big((\sum_{i=1}^{\infty} b_i - \frac{2\delta}{3})_+\big) < \infty & \text{if } \tau \in F(B)' \\ d_\tau\big((\sum_{i=1}^{\infty} b_i - \frac{2\delta}{3})_+\big) = \infty & \text{if } \tau \in F(B). \end{array}$$

All the conditions of Lemma 5.2 being satisfied, we have

$$(\sum_{i=1}^{\infty} a_i - 5\epsilon)_+ \preceq (\sum_{i=1}^{\infty} b_i - \frac{\delta}{3})_+.$$

Since

$$(A-6\epsilon)_+ \preceq (\sum_{i=1}^{\infty} a_i - 5\epsilon)_+ \quad \text{and} \quad (\sum_{i=1}^{\infty} b_i - \frac{\delta}{3})_+ \preceq B$$

it follows that $(A-6\epsilon)_+ \preceq B$ for every $\epsilon > 0$, and hence $A \preceq B$. $\square$

6. Quasicontinuous scale

Kucerovsky and Perera introduced in [19] the notion of quasicontinuous scale for simple C*-algebras of real rank zero in terms of quasitraces. We adapt their definition to our setting.

Definition 6.1. *Let $\mathcal{A}$ be a C*-algebra with non-empty tracial simplex $\mathcal{T}(\mathcal{A})$. The lower semicontinuous affine function $S := \widehat{1_{\mathcal{M}(\mathcal{A})}}$ is called the scale of $\mathcal{A}$. The scale S is said to be quasicontinuous if:*

(i) *the set $F_\infty := \{\tau \in \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A})) \mid S(\tau) = \infty\}$ is finite (possibly empty);*
(ii) *the complementary face F'_∞ of $\mathrm{co}(F_\infty)$ is closed (possibly empty);*
(iii) *the restriction $S\mid_{F'_\infty}$ of the scale S to F'_∞ is continuous.*

Notice that while the scale function S depends on the normalization chosen for $\mathcal{T}(\mathcal{A})$, the quasicontinuity of S does not. Indeed, let e, f be two positive elements in $\mathrm{Ped}(\mathcal{A})$ and S_e and S_f be the scales relative to $\mathcal{T}(\mathcal{A})_{e\mapsto 1}$ and $\mathcal{T}(\mathcal{A})_{f\mapsto 1}$ respectively (see 2.3). Let ψ be the homeomorphism

$$\psi : \mathcal{T}(\mathcal{A})_{e\to 1} \mapsto \mathcal{T}(\mathcal{A})_{f\mapsto 1} \quad \text{given by } \psi(\tau) := \frac{1}{\tau(f)}\tau.$$

Then

$$S_f(\psi(\tau)) = \frac{S_e(\tau)}{\hat{f}(\tau)} \quad \forall\, \tau \in \mathcal{T}(\mathcal{A})_{e\mapsto 1}\,.$$

Since $f \in \mathrm{Ped}(\mathcal{A})$, by the definition of the topology on $\mathcal{T}(\mathcal{A})$, $\hat{f}$ is a continuous function on $\mathcal{T}(\mathcal{A})$ which by the simplicity of $\mathcal{A}$ never vanishes, thus $\frac{1}{\hat{f}(\tau)}$ is continuous. Furthermore, as stated in §2.2, ψ maps faces onto faces, thus if S_e satisfies conditions (i)-(iii) so does S_f. Because of this, we can drop the reference to the specific normalization used and just refer to a scale S.

We can view C*-algebras with quasicontinuous scale as a natural generalization of C*-algebras with finite extremal boundary and extend to them the proof of Theorem 5.4 of the previous section.

For that purpose, notice first that if $\mathcal{A}$ has quasicontinuous scale S and B is a positive element of $\mathcal{M}(\mathcal{A})_+$, then the face $F(B) = \mathrm{co}\{\tau \in \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\}$ (see 5.1) is contained in $\mathrm{co}\, F_\infty$ and more precisely,

$$F(B) = \mathrm{co}\{\tau \in F_\infty \mid B \notin I_\tau\}. \tag{6.1}$$

As a consequence,

$$F(B)' = \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty, \tag{6.2}$$

that is $\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \cap F'_\infty = \emptyset$ and every element in the face $F(B)'$ that is not in $\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \cup F'_\infty$ is a unique convex combination of two elements in those faces. Both terms in this direct convex sum are closed and hence so is $F(B)'$.

We start with the following lemmas.

Lemma 6.2. *Let $\mathcal{A}$ be a simple C*-algebra, $K \subset \mathcal{T}(\mathcal{A})$ a closed set, and $A \le B \in \mathcal{M}(\mathcal{A})_+$. If $\hat{B}\mid_K$ is continuous, then $\hat{A}\mid_K$ too is continuous.*

Proof. $\hat{B}\mid_K = \hat{A}\mid_K + \widehat{B-A}\mid_K$ and since the first function is continuous and the second two functions are lower semicontinuous, it is immediate to see that both must be continuous. ∎

Lemma 6.3. *Let $\mathcal{A}$ be a σ-unital simple C*-algebra, $K \subset \mathcal{T}(\mathcal{A})$ a closed set, $A, B \in \mathcal{M}(\mathcal{A})_+$ with $\hat{A}\mid_K$ continuous, and assume that*

$$d_\tau(A) < d_\tau(B) \quad \text{for all } \tau \in K \text{ for which } d_\tau(B) < \infty.$$

Then for every $\epsilon > 0$, there exist $\delta > 0$ and $\alpha > 0$ such that

$$d_\tau((A-\epsilon)_+) + \alpha \le d_\tau((B-\delta)_+) \quad \forall\ \tau \in K.$$

If furthermore $\hat{B}\mid_K$ is continuous, then $d_\tau((B-\delta)_+) < \infty$ for all $\tau \in K$.

Proof. Assume without loss of generality that $\|A\| = 1$ and let f_ϵ be the function defined in (2.1). Then

$$\chi_{(\epsilon,1]}(t) \le f_\epsilon(t) \le \min\{1, \frac{t}{\epsilon}\},$$

and hence

$$R_{(A-\epsilon)_+} \le f_\epsilon(A), \tag{6.3}$$

$$f_\epsilon(A) \le \frac{1}{\epsilon}A, \tag{6.4}$$

$$f_\epsilon(A) \le R_A. \tag{6.5}$$

From (6.3) we have

$$\widehat{f_\epsilon(A)}(\tau) \ge d_\tau\Big((A-\epsilon)_+\Big) \quad \forall \tau \in \mathcal{T}(\mathcal{A})\,. \tag{6.6}$$

From (6.4) and Lemma 6.2 it follows that $\widehat{f_\epsilon(A)}\mid_K$ is continuous. From (6.5) it follows that

$$\widehat{f_\epsilon(A)}(\tau) \le \tau(R_A) = d_\tau(A) \le d_\tau(B),$$

the last inequality being strict when $d_\tau(B) < \infty$. As a consequence, the function $\big(d_\tau(B) - \widehat{f_\epsilon(A)}(\tau)\big)\mid_K$ is strictly positive lower semicontinuous, and hence it has a minimum 2α on K.

Let $B_n = (B - \frac{1}{n})_+$. Then $0 \le B_n \uparrow B$ (in norm) and hence $d_\tau(B_n) \uparrow d_\tau(B)$ for every $\tau \in \mathcal{T}(\mathcal{A})$. Since all the functions $d_\tau(B_n)$ are lower semicontinuous, by the compactness of K there is an n such that,

$$d_\tau(B_n) \ge \widehat{f_\epsilon(A)}(\tau) + \alpha \quad \forall \tau \in K.$$

Thus for $\delta := \frac{1}{n}$ we have

$$d_\tau((B-\delta)_+) \ge \widehat{f_\epsilon(A)} + \alpha \ge d_\tau((A-\epsilon)_+) + \alpha \quad \forall \tau \in K,$$

where the last inequality follows from (6.6).

If in addition $\hat{B}$ is continuous on K, then by the same reasoning as for A, for every $\delta > 0$ we have

$$d_\tau\big((B-\delta)_+\big) \le \widehat{f_\delta(B)}(\tau) \le \frac{1}{\delta}\hat{B}(\tau).$$

Thus $d_\tau\big((B-\delta)_+\big) < \infty$ for every $\tau \in K$. $\square$

Lemma 6.4. *Let $\mathcal{A}$ be a σ-unital nonunital simple C*-algebra, $P \in \mathcal{M}(\mathcal{A})$ be a projection, $K \subset \mathcal{T}(\mathcal{A})$ be a closed set such that $\widehat{P}\mid_K$ is continuous, and let $A := \sum_{n=1}^\infty A_n$ be the strictly converging sum of elements $A_n \in (P\mathcal{M}(\mathcal{A})P)_+$. Assume furthermore that there exists an approximate identity $\{e_n\}_{n=1}^\infty$ for $P\mathcal{A}P$ with $e_{n+1}e_n = e_n$ for all $n \in \mathbb{N}$ such that for all $m \ge 1$, there exists $N \in \mathbb{N}$ with $e_m \sum_{j=N}^\infty A_j = 0$. Then for every $\delta \ge 0$,*

(i) $d_\tau\Big(\Big(\sum_{j=n}^\infty A_j - \delta\Big)_+\Big) \to 0$ *uniformly on* K.

(ii) $d_\tau\Big(\Big(\sum_{j=1}^n A_j - \delta\Big)_+\Big) \to d_\tau\Big(\Big(A - \delta\Big)_+\Big)$ *uniformly on* K.

Proof. Assume without loss of generality that $\|\sum_{j=1}^\infty A_j\| \leq 1$ and let $\epsilon > 0$ be given.

(i) Since $d_\tau\Big(\big(\sum_{j=n}^\infty A_j - \delta\big)_+\Big) \leq d_\tau\big(\sum_{j=n}^\infty A_j\big)$ for every n by Lemma 2.3, it is enough to prove the statement for $\delta = 0$.

Since e_n has a local unit, it belongs to the Pedersen ideal and hence by the definition of the topology on $\mathcal{T}(\mathcal{A})$, $\widehat{e_n}$ is continuous. As $\widehat{e_n} \uparrow \widehat{P}$, and $\widehat{P}\mid_K$ is continuous, by Dini's theorem the convergence is uniform on K. Thus choose m such that $0 \leq \widehat{P} - \widehat{e_{m-1}} < \epsilon$ on K. Now choose N such that

$$e_m \sum_N^\infty A_n = 0$$

Then for every $n \geq N$

$$\sum_n^\infty A_n = (P - e_m)\Big(\sum_n^\infty A_n\Big)(P - e_m) \leq (P - e_m)^2 \leq P - e_m.$$

Since

$$R_{\sum_n^\infty A_n} \leq R_{P-e_m} \leq P - e_{m-1},$$

because $(P - e_{m-1})(P - e_m) = (P - e_m)$ we thus have for every $\tau \in K$ that

$$d_\tau\Big(\sum_n^\infty A_n\Big) \leq \tau(P - e_{m-1}) < \epsilon,$$

which proves (i).

(ii) By Lemma 2.3 and Lemma 2.4 (iii) we have for all $n \geq 1$ and $\tau \in K$, that

$$\begin{aligned} d_\tau\Big(\big(\sum_{j=1}^n A_j - \delta\big)_+\Big) &\leq d_\tau\Big(\big(\sum_{j=1}^\infty A_j - \delta\big)_+\Big) \\ &\leq d_\tau\Big(\big(\sum_{j=1}^n A_j - \delta\big)_+\Big) + d_\tau\Big(\sum_{j=n+1}^\infty A_j\Big). \end{aligned}$$

Thus (ii) follows from (i). □

Remark 6.5.

(i) *The condition that for every* n *there exists an* $N \in \mathbb{N}$ *such that* $e_n \sum_{j=N}^\infty a_j = 0$ *cannot be removed. Consider for instance an element* $b \in \mathcal{A}_+$ *such that* $R_b = P$ *and let* $a_n := \frac{1}{2^n} f_{1/n}(b)$. *Then* $\sum_1^\infty a_n$ *converges in norm, hence strictly, but since* $R_{\sum_n^\infty a_n} = R_b$ *for all* n, *it follows that* $d_\tau\Big(\sum_n^\infty a_n\Big) \not\to 0$.

(ii) *Substituting the continuity of* $\hat{P}\mid_K$ *with the (weaker) condition of the continuity of* $\hat{A}\mid_K$ *still permits to obtain uniform convergence on* K *for* $\delta > 0$ *but not for* $\delta = 0$. *Indeed consider the case of* $\mathcal{A} := \mathcal{B} \otimes \mathcal{K}$ *with* $\mathcal{B}$ *unital and simple,* $K = \mathcal{T}(\mathcal{A})$, $P = 1_{\mathcal{M}(\mathcal{A})}$, $A_k := \frac{1}{2^k} 1_{\mathcal{B}} \otimes e_{k,k}$, $A := \sum_1^\infty A_k$. *Then* $\hat{A}(\tau) = 1$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$, *hence it is continuos, and* $1_{\mathcal{B}} \otimes e_{m,m} \sum_{m+1}^\infty A_k = 0$ *for all* m *but* $d_\tau\big(\sum_n^\infty A_k\big) = \infty$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$ *and* $n \in \mathbb{N}$.

Theorem 6.6. *Let $\mathcal{A}$ be a σ-unital simple C*-algebra with strict comparison of positive elements by traces and with quasicontinuous scale. Then strict comparison of positive element by traces holds in $\mathcal{M}(\mathcal{A})$.*

Proof. In following the proof of Theorem 5.4, let $\epsilon > 0$, $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $B \notin \mathcal{A}$, $A \in I(B)$, and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$.

Since $A, B \leq 1_{\mathcal{M}(\mathcal{A})}$ and $S\mid_{F'_\infty}$ is continuous, by Lemma 6.2 the functions $\hat{A}\mid_{F'_\infty}$ and $\hat{B}\mid_{F'_\infty}$ are also continuous. Thus by Lemma 6.3 there are $\delta' > 0$ and $\alpha > 0$ such that for all $\tau \in F'_\infty$

$$d_\tau((A-\epsilon)_+) + \alpha \leq d_\tau((B-\delta')_+) < \infty.$$

Moreover, for all $\tau \in F_\infty$ for which $B \in I_\tau$, we can find $\delta_\tau > 0$, $\alpha_\tau > 0$ such that

$$d_\tau((A-\epsilon)_+) + \alpha_\tau \leq d_\tau((B-\delta_\tau)_+) < \infty.$$

Let $\alpha = \min\{\alpha', \alpha_\tau \mid \tau \in F_\infty,\, B \in I_\tau\}$ and $\delta = \min\{\delta', \delta_\tau \mid \tau \in F_\infty,\, B \in I_\tau\}$. Since $F(B)' = \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$ (see 5.2), it thus follows that

$$d_\tau((A-\epsilon)_+) + \alpha \leq d_\tau((B-\delta)_+) < \infty \quad \forall\, \tau \in F(B)'.$$

By (5.1), $F(B) = \mathrm{co}\{\tau \in F_\infty \mid B \notin I_\tau\}$ hence we can find $\delta_o > 0$ such that $d_\tau\big((B-\delta_o)_+\big) = \infty$ for all $\tau \in F(B)$. By replacing if necessary δ with $\min\{\delta_o, \delta\}$, we see that the condition (5.15) in the beginning of the proof of Theorem 5.4 is satisfied.

Thus we proceed exactly as in the proof of Theorem 5.4 and decompose A and B into bidiagonal series with "small" remainders:

$$A = \sum_{i=1}^{\infty} a_i + a_o \quad \text{and} \quad B = \sum_{i=1}^{\infty} b_i + b_o.$$

Recall that from Theorem 4.2, the bidiagonal series can be chosen so that for every $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ for which $e_n \sum_N^\infty a_k = e_n \sum_N^\infty b_k = 0$ for some approximate identity $\{e_n\}$ satisfying the condition $e_{n+1}e_n = e_n$ for all n. Then we obtain as in the proof of Theorem 5.4 that the conditions (i)-(iv) of Lemma 5.3 are satisfied for $F = F(B)$ and $F' = F(B)'$. Condition (v) holds since $F \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A})) \subset F_\infty$. The convergence in (vi) and (vii) is pointwise by Corollary 3.4, hence it is uniform on $\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\}$ because this face has finite extremal boundary. In view of the decomposition in (5.2) $F(B)' = \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$, we see that uniform convergence on $F(B)'$ holds if it holds on F'_∞. Lemma 6.4 applied to $P = 1_{\mathcal{M}(\mathcal{A})}$ and $K = F'_\infty$ guarantees this uniform convergence. Thus all the condition of Lemma 5.3 are satisfied and the rest of the proof of Theorem 5.4 applies without changes. □

7. Positive linear combinations of projections

It is well known that every element of $B(H)$ is a linear combination of projections. The same property holds for all von Neumann algebras without a finite type I direct summand with infinite dimensional center [8]. However this property may fail even for C*-algebras of real rank zero (see [16, Proposition 5.1]).

In the process of investigating linear combination of projections in C*-algebras, we found it convenient to consider the following stronger condition:

Definition 7.1. *A C*-algebra $\mathcal{A}$ has universal constant V if every selfadjoint element a in $\mathcal{A}$ is a linear combination of N projections $p_j \in \mathcal{A}$ with $a = \sum_1^N \lambda_j p_j$ for some $N \in \mathbb{N}$, and $\lambda_j \in \mathbb{R}$, satisfying the condition*

$$\sum_1^N |\lambda_j| \leq V\|a\|.$$

If furthermore N can be chosen independently of the element a, we say that $\mathcal{A}$ has universal constants V and N.

C*-algebras that have such universal constants include:

- von Neumann algebras without a finite type I summand with infinite dimensional center [8];
- unital properly infinite C*-algebras ([11, Propositions 2.6, 2.7]);
- unital simple separable C*-algebras with real rank zero, stable rank one, strict comparison of projections and finite extremal tracial boundary ([16, Theorem 4.4]);
- corners $P\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,P$ with$P$ projection in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ of unital simple separable C*-algebras with real rank zero, stable rank one, strict comparison of projections and finite extremal tracial boundary ([17, Theorem 4.4]).

A linear combination $A = \sum_1^n \alpha_j p_j$ with projections $p_j \in \mathcal{A}$ and $\alpha_j > 0$ will be called a *positive linear combination of projections* or PCP for short.

In [11, Proposition 2.7] we proved that if a C*-algebra $\mathcal{A}$ has such universal constants and if furthermore $\mathcal{A}_+$ is the closure of PCPs in $\mathcal{A}$ then every positive invertible element of $\mathcal{A}$ is a PCP.

Thus if both conditions hold for all corners $p\mathcal{A}p$ of $\mathcal{A}$, then all positive locally invertible elements are PCP. A key tool for the further investigation of PCP elements was the fact that a direct sum of projection and of a "small" positive perturbation is also PCP ([11, Lemma 2.2]).

We can obtain the following result under less restrictive conditions:

Lemma 7.2. *Let $\mathcal{A}$ be a C*-algebra, $p \in \mathcal{A}$ be a projection such that the corner algebra $p\mathcal{A}p$ has universal constant V. Then*

(i) *$p + b$ is a PCP for every $b = b^* \in p\mathcal{A}p$ with $\|b\| \leq \frac{1}{V}$. If the corner algebra $p\mathcal{A}p$ has universal constants V and N, the number of projections needed in the PCP is $N + 1$.*

(ii) *$p+b$ is a PCP for every $b \in \mathcal{A}_+$ with $b = qb = bq$ for some projection $q \in \mathcal{A}$ such that $q \perp p$ with $q \prec p$ and $\|b\| \leq \frac{1}{1+V}$. Furthermore, if $p\mathcal{A}p$ has universal constants V and N, $p + b$ can be decomposed as a PCP of $N + 4$ projections.*

(iii) *$p + b$ is a PCP for every $b \in \mathcal{A}_+$ with $b = qb = bq$ for some projection $q \in \mathcal{A}$ such that $q \perp p$ with $m[q] \leq [p]$ for some $m \in \mathbb{N}$ with $m \geq \|b\|(1 + V)$.*

Proof.

(i) By hypothesis we can find N real numbers λ_j and projections $q_j \in \mathcal{A}$ with $q_j \leq p$ such that $b = \sum_{j=1}^N \lambda_j q_j$ and $\sum_{j=1}^N |\lambda_j| \leq V\|b\| \leq 1$.

$$p + b = \sum_{\lambda_j \geq 0} \lambda_j q_j + \sum_{\lambda_j < 0} (-\lambda_j)(p - q_j) + \big(1 + \sum_{\lambda_j < 0} \lambda_j\big)p$$

is a PCP of $N + 1$ projection.

(ii) Assume without loss of generality that $b \neq 0$ and hence $V\|b\| < 1$ and let $\beta := \frac{1}{1-V\|b\|}$. Then $1 < \beta \leq \frac{1}{\|b\|}$. Following the proof of [11, Lemma 2.9], let $v \in \mathcal{A}$ be a partial isometry such that $v^*v = q$ and $vv^* = p' \leq p$. Define

$$r_1 := \beta b + v\sqrt{\beta b - (\beta b)^2} + \sqrt{\beta b - (\beta b)^2}v^* + p' - \beta vbv^*$$
$$r_2 := \beta b - v\sqrt{\beta b - (\beta b)^2} - \sqrt{\beta b - (\beta b)^2}v^* + p' - \beta vbv^*$$

Then r_1 and r_2 are projections in $\mathcal{A}$ and $\beta b = \frac{1}{2}(r_1 + r_2) - p' + \beta vbv^*$, hence

$$p + b = \frac{1}{2\beta}(r_1 + r_2) + \frac{1}{\beta}(p - p') + (1 - \frac{1}{\beta})\Big(p + \frac{vbv^*}{1 - \frac{1}{\beta}}\Big)$$

Now $0 \leq \frac{vbv^*}{1-\frac{1}{\beta}} \in p'\mathcal{A}p' \subset p\mathcal{A}p$ and $\|\frac{vbv^*}{1-\frac{1}{\beta}}\| = \frac{\|b\|}{1-\frac{1}{\beta}} = \frac{1}{V}$. Then by part (i) $p + \frac{vbv^*}{1-\frac{1}{\beta}}$ is a PCP, and hence so is $p + b$. Furthermore, if $p\mathcal{A}p$ has universal constants V and N, by part (i), $p + \frac{vbv^*}{1-\frac{1}{\beta}}$ can be decomposed as a PCP of $N + 1$ projections and hence $p + b$ can be decomposed as a PCP of $N + 4$ projections.

(iii) Decompose $p = \bigoplus_{i=1}^m p_i$ into projections $p_i \in \mathcal{A}$ with $q \prec p_i$ for each i. Then

$$p + b = \sum_{i=1}^m \big(p_i + \frac{1}{m}b\big).$$

For each i it follows from part (ii) that $p_i + \frac{1}{m}b$ is a PCP and hence so is $p + b$. □

Our next lemma permits us to embed isomorphically σ-unital hereditary subalgebras of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ into unital corners of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ *with control on the "size" of the corner*. When $B \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ we use the following notations:
$\mathrm{her}(B) := \overline{B(\mathcal{A} \otimes \mathcal{K})B^*}$ hereditary subalgebra of $\mathcal{A} \otimes \mathcal{K}$
$\mathrm{Her}(B) = \overline{B\,\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\,B^*}$ hereditary subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

Lemma 7.3. *Let $\mathcal{A}$ be a C^*-algebra and $B \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+$ be such that the hereditary algebra $\mathrm{her}(B)$ of $\mathcal{A} \otimes \mathcal{K}$, has an approximate unit $\{f_j\}$ consisting of an increasing sequence of projections. Then there is a partial isometry $W \in (\mathcal{A} \otimes \mathcal{K})^{**}$ such that*
(i) $W^*W = R_B$
(ii) $WW^* \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$
(iii) $WB \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$
(iv) $W\,\mathrm{Her}(B)W^* \subseteq R\,\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\,R$ *where* $R := WW^*$.
(v) *The onto map*

$$\mathrm{Her}(B) \ni X \to \Phi(X) := WXW^* \in \mathrm{Her}(\Phi(B))$$

*is a trace preserving *-isomorphism of hereditary algebras.*

Proof. Let $e_j := f_j - f_{j-1}$ (with $f_0 := 0$) and let $I_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = \sum_1^\infty E_j$ be decomposition of the identity into projections $E_j \sim I_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. As $e_j \preceq E_j$ there are partial isometries $v_j \in \mathcal{A} \otimes \mathcal{K}$ such that $v_j^*v_j = e_j$ and $v_jv_j^* \leq E_j$. Let $W := \sum_1^\infty v_j$. The series converges in the strong topology of $(\mathcal{A} \otimes \mathcal{K})^{**}$ because both the range projections of the partial isometries v_j are mutually orthogonal and so are the range projections of v_j^*. Then

$$W^*W = \sum_1^\infty e_j = \lim_j f_j = R_B$$

and the convergence is again in the strong topology of $(\mathcal{A}\otimes\mathcal{K})^{**}$. On the other hand, $WW^*=\sum_1^\infty v_jv_j^*$ in the strict topology. Thus the projection $R:=WW^*$ belongs to $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Next we show that $WB\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Let $a\in\mathcal{A}\otimes\mathcal{K}$. Then $Baa^*B\in \mathrm{her}(B)$, hence $f_kBa\to Ba$ in norm, or equivalently $\sum_1^n e_jBa$ converges in norm to Ba. Since $We_j=v_j$ for all j, we have

$$W\sum_1^n e_jBa=\sum_1^n v_jBa\to WBa\in\mathcal{A}\otimes\mathcal{K}$$

since the convergence is in norm. On the other hand, since $\sum_1^\infty v_jv_j^*$ converges strictly, $\|a\sum_n^\infty v_jv_j^*\|\to 0$ for $n\to\infty$, hence

$$\|a\sum_n^\infty v_jv_j^*W\|=\|a\sum_n^\infty v_j\|\to 0$$

and thus $aW\in\mathcal{A}\otimes\mathcal{K}$, whence $aWB\in\mathcal{A}\otimes\mathcal{K}$.

This concludes the proof of (i)-(iii).

Next, $BW^*\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, hence $WB\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,BW^*\subset\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ and hence $W\,\mathrm{Her}(B)W^*\subseteq R\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,R$, i.e., (iv) holds. Finally, proving (v) is routine. □

Remark 7.4.

(i) *The above result can be seen as the construction of a projection $P\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ that is equivalent to the open projection R_B in the sense of Peligrad and Zsido* [26] *(see also* [23]*).*

(ii) *$\mathcal{A}$ has real rank zero if and only if every hereditary subalgebra of $\mathcal{A}$ has an approximate identity of projections (*[3]*).*

Proposition 7.5. *Let $\mathcal{A}$ be a simple separable C^*-algebra with real rank zero, stable rank one, strict comparison of projections, and finite extremal boundary. Let $P\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\setminus\mathcal{A}\otimes\mathcal{K}$ be a projection. Then $P+B$ is a PCP for every $B\in(P^\perp\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,P^\perp)_+$ such that $\tau(R_B)<\infty$ for all those $\tau\in\mathcal{T}(\mathcal{A})$ for which $\tau(P)<\infty$.*

Proof. Let $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))=\{\tau_j\}_1^n$ and notice that $F(B)'=\{\tau\in\mathcal{T}(\mathcal{A})\mid\tau(P)<\infty\}$. Since $\mathcal{A}$ has real rank zero and R_B is an open projection, it has a decomposition $R_B=\bigoplus_1^\infty r_j$ into a strictly converging sum of mutually orthogonal projections $r_j\in\mathcal{A}\otimes\mathcal{K}$. By [17, Theorem 5.1], $P\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,P$ has a universal constants V and let $m>\|B\|(1+V)$ be an integer. Since $\tau(\bigoplus_1^\infty r_j)<\infty$ for all those $\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ for which $\tau(P)<\infty$ and there are only finitely many extremal traces, there exists a k such that $\tau(\bigoplus_k^\infty r_j)<\frac{1}{m}\tau(P)$ for all $\tau\in F(B)'$. Let

$$B':=B^{1/2}(\bigoplus_1^{k-1}r_j)B^{1/2}\quad\text{and}\quad B'':=B-B'=B^{1/2}(\bigoplus_k^\infty r_j)B^{1/2}$$

Then $B'\in\mathcal{A}\otimes\mathcal{K}_+$ and $B''\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})_+$. Moreover,

$$R_{B'}\preceq\bigoplus_1^{k-1}r_j\quad\textit{and}\quad R_{B''}\preceq\bigoplus_k^\infty r_j$$

where the Murray-von Neumann subequivalence $\preceq$ is in $(\mathcal{A}\otimes\mathcal{K})^{**}$. Thus

$$\tau(R_{B'})<\infty\quad\forall\tau\in\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\quad\text{and}\quad\tau(R_{B''})<\frac{1}{m}\tau(P)\quad\forall\tau\in F(B)'.$$

By [16, Theorem 6.1], B' is a PCP. Thus it remains to prove that $P + B''$ is also a PCP.

By Lemma 7.3 there is a partial isometry $W \in (\mathcal{A} \otimes \mathcal{K})^{**}$ with $W^*W = R_{B''}$, $R := WW^* \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ and that induces an isomorphism

$$\mathrm{Her}(B'') \ni X \to \Phi(X) := WXW^* \in \mathrm{Her}(\Phi(B'')) \subset R\,\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\,R.$$

Notice that $R \sim R_{B''}$ in $(\mathcal{A} \otimes \mathcal{K})^{**}$ and hence

$$\tau(R) = \tau(R_{B''}) \leq \tau(R_B) = \tau(\bigoplus_1^\infty r_j) \leq \tau(P^\perp) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A})\,.$$

Thus if $\tau(P^\perp) < \infty$, then $\tau(\bigoplus_1^\infty r_j) < \infty$, and hence

$$\tau(R_{B''}) \leq \tau(\bigoplus_k^\infty r_j) < \tau(P^\perp).$$

By strict comparison of projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ (see [17, Theorem 3.2], or a consequence of Theorem 5.4) it follows that $R \preceq P^\perp$. Without loss of generality we can assume that $R \leq P^\perp$. Now let $W' := P \oplus W$. Then the map

$$\mathrm{Her}(P \oplus B') \ni X \to \Phi'(X) = WXW^* \in \mathrm{Her}\left(\Phi'(P \oplus B')\right)$$

is a trace preserving *-isomorphism. Now

$$\Phi'(P \oplus B'') = P \oplus \Phi(B'') \quad \text{and} \quad \Phi(B'') = R\Phi(B'')R.$$

Furthermore, $\tau(R) < \frac{1}{m}\tau(P)$ for all $\tau \in F(B)'$ and since $P \notin \mathcal{A} \otimes \mathcal{K}$, by the strict comparison of projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, $m[R] \leq [P]$. Then $\Phi(P \oplus B'')$ is a PCP by Lemma 7.2 (iii). Since Φ' is an isomorphism of hereditary algebras, $P + B''$ is also a PCP and hence so is $P + B$. □

Next we need a result on principal ideals. While the structure of two-sided norm closed ideals of $\mathcal{M}(\mathcal{A})$ is difficult to analyze in general, a case where this structure is well understood is the following.

Theorem 7.6. [27, Theorem 4.4, Proposition 4.1] *Assume that $\mathcal{A}$ is a simple unital non elementary C^*-algebra with strict comparison of positive elements by traces and with finite extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$. Then*

(i) *A proper ideal $\mathcal{J}$ of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is maximal if and only if $\mathcal{J} = I_\tau$ for some τ in $\partial_e(\mathcal{T}(\mathcal{A}))$.*

(ii) *Every proper ideal of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ that properly contains $\mathcal{A} \otimes \mathcal{K}$ is the intersection of a (finite) number of maximal ideals. Therefore there are exactly $2^n - 1$ proper ideals of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ properly containing $\mathcal{A} \otimes \mathcal{K}$.*

As a consequence if $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus \mathcal{A} \otimes \mathcal{K}$ then

$$I(T) = \cap\{I_\tau \mid \tau \in F(T)'\} = \cap\{I_\tau \mid \tau \in F(T)' \cap \partial_e(\mathcal{T}(\mathcal{A}))\}. \tag{7.1}$$

Proposition 7.7. *Let $\mathcal{A}$ be a simple unital C^*-algebra with real rank zero strict comparison of positive elements by traces and with finite extremal boundary and let $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus \mathcal{A} \otimes \mathcal{K}$. Then there is a $\delta > 0$ such that*

(i) *$I(T) = I\big((T - \delta)_+\big)$;*

(ii) *there is a projection P such that $I(P) = I(T)$ and $T \geq \delta P$.*

Proof. The case when $\mathcal{A} \otimes \mathcal{K} = \mathcal{K}$ and hence $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) = B(H)$ follows from standard operator theory, so assume without loss of generality that $\mathcal{A}$ is not elementary.

(i) By (2.7), for every $\tau \in \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ for which $T \notin I_\tau$ (that is, for every τ in $F(T) \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$,) there is a $\delta_\tau > 0$ such that $(T - \delta_\tau)_+ \notin I_\tau$. Let

$$\delta := \min\{\delta_\tau \mid F(T) \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))\}.$$

Then $(T-\delta)_+ \notin I_\tau$ for all $\tau \in F(T) \cap \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$, hence $F(T) \subset F\big((T-\delta)_+\big)$. On the other hand, $(T-\delta)_+ \leq T \in I_\tau$ for all $\tau \in F(T)'$, hence $F\big((T-\delta)_+\big)' \subset F(T)'$. Thus $F\big((T-\delta)_+\big)' = F(T)'$ and by (7.1) $I(T) = I\big((T-\delta)_+\big)$.

(ii) By (i), $I((T-\delta)_+) = I(T)$, and hence

$$d_\tau((T-\delta)_+) \begin{cases} < \infty & \tau \in F(T)' \\ = \infty & \tau \in F(T). \end{cases}$$

By [17, Proposition 3.3] there is a projection $Q \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \setminus \mathcal{A} \otimes \mathcal{K}$ such that $\tau(Q) = \frac{1}{2} d_\tau\big((T-\delta)_+\big)$ for all $\tau \in \partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ and hence for all $\tau \in \mathcal{T}(\mathcal{A})$. Then $F(Q)' = F(T)'$ and hence by (7.1), $I(Q) = I(T)$. Now by strict comparison of positive elements in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ (Theorem 5.4) it follows that $Q \preceq (T-\delta)_+$. Thus by Lemma 2.5, there is a projections $P\,\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\,P$ such that $T \geq \delta P$ and $P \sim Q$, and hence $I(P) = I(Q) = I(T)$. □

We list here a property we will need in the proof our the next theorem

Lemma 7.8. *Let $\mathcal{B}$ be a C*-algebra. Every $g \in C([0,1])$ is uniformly continuous on the positive part of the unit ball of $\mathcal{B}$.*

Proof. Let a, b be in the unit ball of $\mathcal{B}$ and let $\epsilon > 0$. Find a polynomial p_n such that $\|g - p_n\|_\infty < \frac{\epsilon}{3}$. Then $\|g(a) - p_n(a)\| < \frac{\epsilon}{3}$ and $\|g(b) - p_n(b)\| < \frac{\epsilon}{3}$. Moreover, $\|p_n(a) - p_n(b)\| \leq c\|a-b\|$ where $p_n(t) = \sum_0^n \alpha_j t^j$ and $c = \sum_1^n j|\alpha_j|$. Indeed, since

$$\|a^n - b^n\| \leq \|a^{n-1}\|\|a-b\| + \|a^{n-2}\|\|b\|\|a-b\| + \cdots + \|b^{n-1}\|\|\|a-b\| \leq n\|a-b\|$$

and hence

$$\|p_n(a) - p_n(b)\| = \|\sum_1^n \alpha_j(a^j - b^j)\| \leq \|\sum_1^n |\alpha_j|\|a^j - b^j\| \leq \|\sum_1^n j|\alpha_j|\|a-b\|$$

Set $\delta = \frac{\epsilon}{3c}$. For every $\|a-b\| < \delta$ it follows that $\|p_n(a) - p_n(b)\| < \frac{\epsilon}{3}$. Thus $\|g(a) - g(b)\| < \epsilon$. □

Theorem 7.9. *Let $\mathcal{A}$ be a simple separable C*-algebra with real rank zero, stable rank one, strict comparison of projections, and finite extremal boundary, and let $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+$. Then T is a PCP if and only if $\tau(R_T) < \infty$ for all $\tau \in F(T)'$, that is, for all τ for which $T \in I_\tau$.*

Proof. We first prove the necessity. Assume that $T = \sum_{j=1}^n \lambda_j P_j$ for some $\lambda_j > 0$ and projections $P_j \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ and that $T \in I_\tau$ for some $\tau \in \mathcal{T}(\mathcal{A})$. Since $\lambda_j P_j \leq T$, it follows that $P_j \in I_\tau$ and thus $\tau(P_j) < \infty$. Let $R = \bigvee_1^n P_j \in (\mathcal{A} \otimes \mathcal{K})^{**}$. Since

$$\tau(R_T) \leq \tau(R) \leq \sum_1^n \tau(P_j)$$

we conclude that $\tau(R_T) < \infty$.

Now we prove the sufficiency. If $T \in \mathcal{A} \otimes \mathcal{K}$, the result is proven in [16, Theorem 6.1]. Thus assume that $T \notin \mathcal{A} \otimes \mathcal{K}$ and further that $\|T\| \leq 1$. By Proposition 7.7, there is a $0 < \delta < 1$ and a projection $P \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ for which $I(P) = I(T)$

and $T \geq \delta P$. Assume further that $\delta < \frac{6}{7}$. Since $P \notin \mathcal{A} \otimes \mathcal{K}$, by [33], P can be decomposed into the sum $P = P_1 + P_2$ of two projections $P_1 \sim P_2$. Then for $i = 1, 2$

$$\tau(P_i) = \begin{cases} < \infty & \tau \in F(T)' \\ = \infty & \tau \in F(T) \end{cases}$$

and hence $I(P_1) = I(P_2) = I(T)$. Set now

$$T' = T - \frac{\delta}{2} P_1 = T - \delta P + \frac{\delta}{2} P_1 + \delta P_2.$$

Since $T = T' + \frac{\delta}{2} P_1$, it is enough to prove that T' is a PCP. Notice that $R_{T'} \leq R_T$. Let f_1 and f_2 be the continuous functions defined by

$$f_1(t) = \begin{cases} t & t \in [0, \frac{2}{3}\delta] \\ 0 & t \in [\frac{5}{6}\delta, 1] \\ \text{linear} & t \in [\frac{2}{3}\delta, \frac{5}{6}\delta] \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0 & t \in [0, \frac{2}{3}\delta] \\ t & t \in [\frac{5}{6}\delta, 1] \\ \text{linear} & t \in [\frac{2}{3}\delta, \frac{5}{6}\delta] \end{cases}$$

Now consider the continuous functions g_1 and g_2 defined by

$$g_1(t) = \begin{cases} 0 & t \in [0, \frac{\delta}{3}] \cup [\frac{2\delta}{3}, 1] \\ \frac{\delta}{2} & t = \frac{\delta}{2} \\ \text{linear} & \text{elsewhere} \end{cases} \quad \text{and} \quad g_2(t) = \begin{cases} 0 & t \in [0, \frac{5\delta}{6}] \cup [\frac{7\delta}{6}, 1] \\ \delta & t = \delta \\ \text{linear} & \text{elsewhere} \end{cases}$$

Then for all $t \in [0, 1]$,

$$f_1(t) + f_2(t) = t \tag{7.2}$$

$$g_1(t) \leq f_1(t) \quad \text{and} \quad g_2(t) \leq f_2(t) \tag{7.3}$$

$$g_1(t) f_2(t) = 0 \quad \text{and} \quad g_2(t) f_1(t) = 0 \tag{7.4}$$

$$f_1(t) \geq \frac{\delta}{3} \quad \text{where } g_1(t) \neq 0 \tag{7.5}$$

$$f_2(t) \geq \frac{5\delta}{6} \quad \text{where } g_2(t) \neq 0. \tag{7.6}$$

Since the functions g_1 and g_2 are both continuous on $[0, 1]$, by Lemma 7.8 they are uniformly continuous on the set of positive contractions. Thus there is an integer n such that $\|g_i(A) - g_i(B)\| \leq \frac{\delta}{4}$ whenever $0 \leq A \leq 1$, $0 \leq B \leq 1$ and $\|A - B\| \leq \frac{1}{n}$.

Reasoning as in the first part of the proof, we can subdivide the projections P_1 and P_2 into an orthogonal sum of n projections

$$P_1 = \sum_1^n P_{1,j} \qquad \text{and} \qquad P_2 = \sum_1^n P_{2,j}$$

such that $I(P_{i,j}) = I(T)$ for all $i = 1, 2$ and $1 \leq j \leq n$. Then

$$T' = \sum_{j=1}^n \left(\frac{1}{n}(T - \delta P) + \frac{\delta}{2} P_{1,j} + \delta P_{2,j} \right)$$

Thus it is enough to prove that for every pair of projections $Q_1 \perp Q_2$, with $Q_i \leq R_T$, and $I(Q_i) = I(T)$ for $i = 1, 2$ we have that the positive element

$$T'' := \frac{1}{n}(T - \delta P) + \frac{\delta}{2} Q_1 + \delta Q_2$$

is a PCP. Notice that $R_{T''} \leq R_T$. Now

$$g_1(\frac{\delta}{2}Q_1 + \delta Q_2) = \frac{\delta}{2}Q_1 \quad \text{and} \quad g_2(\frac{\delta}{2}Q_1 + \delta Q_2) = \delta Q_2.$$

Since $\|\frac{1}{n}(T - \delta P)\| \leq \frac{1}{n}$, it follows that

$$\|g_1(T'') - \frac{\delta}{2}Q_1\| = \|g_1(T'') - g_1(\frac{\delta}{2}Q_1 + \delta Q_2)\| \leq \frac{\delta}{4}.$$

Then $\|\frac{2}{\delta}g_1(T'') - Q_1\| \leq \frac{1}{2}$ and by Lemma 2.1, $\frac{1}{2}Q_1 = (Q_1 - \frac{1}{2})_+ \preceq \frac{2}{\delta}g_1(T'')$. Hence $Q_1 \preceq g_1(T'')$. As a consequence and by (7.5), there is a projection

$$Q_1' \leq R_{g_1(T'')} \leq \frac{1}{\delta/3} f_1(T'') \quad \text{with} \quad Q_1' \sim Q_1 \quad \text{and hence} \quad I(Q_1') = I(T).$$

Similarly, there is a projection

$$Q_2' \leq R_{g_2(T'')} \leq \frac{1}{5\delta/6} f_2(T'') \quad \text{with} \quad Q_2' \sim Q_2 \quad \text{and hence} \quad I(Q_2') = I(T).$$

Notice that $T'' = f_1(T'') + f_2(T'')$ by (7.2). Then

$$T'' = \left(\left(f_1(T'') - \frac{\delta}{3}Q_1'\right) + \frac{5\delta}{6}Q_2'\right) + \left(\left(f_2(T'') - \frac{5\delta}{6}Q_2'\right) + \frac{\delta}{3}Q_1'\right)$$

is a decomposition of T'' into the sum of two positive elements. From (5.10) it follows that $g_1(T'')f_2(T'') = 0$ and hence $Q_2' \perp R_{f_1(T'')}$. Moreover,

$$\tau(R_{f_1(T'')}) \leq \tau(R_T) < \infty$$

for all $\tau \in F(T)'$ and hence for all τ for which $\tau(Q_2') < \infty$. Similarly $Q_1' \perp R_{f_2(T'')}$ and $\tau(R_{f_2(T'')}) < \infty$ for all τ for which $\tau(Q_1') < \infty$. Thus both summands of T'' satisfy the conditions of Proposition 7.5 and hence are a PCP, which concludes the proof.

□

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